COMMON CORE STATE STANDARDS

FOR Mathematics

Contents

Introduction 2
Standards for Mathematical Practice 4
How to Read the Grade Level Standards 6
Overview of the Mathematics Standards, Grades K–5 7
Overview of the Mathematics Standards, Grades 6–8 8
Kindergarten 9
Grade 1 12
Grade 2 15
Grade 3 18
Grade 4 22
Grade 5 26
Grade 6 30
Grade 7 34
Grade 8 38
Introduction to the High School Standards 42
High School—Number and Quantity 43
High School—Algebra 46
High School—Functions 49
High School—Modeling 53
High School—Statistics and Probability 55
High School—Geometry 59
Glossary 63
Appendix: Designing High School Mathematics Courses Based on the Common Core Standards *online at www.corestandards.org*

Introduction
Toward greater focus and coherence

> *The composite standards [of Hong Kong, Korea and Singapore] have a number of features that can inform an international benchmarking process for the development of K–6 mathematics standards in the US. First, the composite standards concentrate the early learning of mathematics on the number, measurement, and geometry strands with less emphasis on data analysis and little exposure to algebra. The Hong Kong standards for grades 1–3 devote approximately half the targeted time to numbers and almost all the time remaining to geometry and measurement.*
>
> Ginsburg, Leinwand and Decker, 2009

> *Mathematics experiences in early childhood settings should concentrate on (1) number (which includes whole number, operations, and relations) and (2) geometry, spatial relations, and measurement, with more mathematics learning time devoted to number than to other topics. The mathematical process goals should be integrated in these content areas. Children should understand the concepts and learn the skills exemplified in the teaching-learning paths described in this report.*
>
> National Research Council, 2009

> *In general, the US textbooks do a much worse job than the Singapore textbooks in clarifying the mathematical concepts that students must learn. Because the mathematics concepts in these textbooks are often weak, the presentation becomes more mechanical than is ideal. We looked at both traditional and non-traditional textbooks used in the US and found this conceptual weakness in both.*
>
> Ginsburg et al., 2005

Notable in the research base for these standards are conclusions from TIMSS and other studies of high-performing countries that the traditional US mathematics curriculum must become substantially more coherent and more focused in order to improve student achievement in mathematics. To deliver on the promise of common standards, the standards must address the problem of a curriculum that is 'a mile wide and an inch deep.' The draft Common Core State Standards for Mathematics are a substantial answer to this challenge.

It is important to recognize that "fewer standards" are no substitute for *focused* standards. Achieving "fewer standards" would be easy to do by simply resorting to broad, general statements. Instead, the draft Common Core State Standards for Mathematics aim for clarity and specificity.

Assessing the coherence of a set of standards is more difficult than assessing their focus. William Schmidt and Richard Houang (2002) have said that content standards and curricula are coherent if they are:

> *articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives. That is, what and how students are taught should reflect* not only the topics *that fall within a certain academic discipline,* but also the key ideas *that determine how knowledge is organized and generated within that discipline. This implies that "to be coherent," a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline. This deeper structure then serves as a means for connecting the particulars (such as an understanding of the rational number system and its properties).* (emphasis added)

The draft Common Core State Standards for Mathematics endeavor to follow such a design, not only by stressing conceptual understanding of the key ideas, but also by continually returning to organizing principles such as place value or the laws of arithmetic to structure those ideas.

The standards in this draft document define what students should understand and be able to do. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between the student who can summon a mnemonic device such as "FOIL" to expand a product such as $(a + b)(x + y)$ and a student who can explain where that mnemonic comes from. Teachers often observe this difference firsthand, even if large-scale assessments in the year 2010 often do not. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding $(a + b + c)(x + y)$. Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

The draft Common Core State Standards for Mathematics begin on the next page with eight Standards for Mathematical Practice. These are not a list of individual math topics, but rather a list of ways in which developing student-practitioners of mathematics increasingly ought to engage with those topics as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years.

Grateful acknowledgment is here made to Dr. Cathy Kessel for editing the draft standards.

Mathematics | Standards for Mathematical Practice

Proficient students of all ages expect mathematics to make sense. They take an active stance in solving mathematical problems. When faced with a non-routine problem, they have the courage to plunge in and try something, and they have the procedural and conceptual tools to continue. They are experimenters and inventors, and can adapt known strategies to new problems. They think strategically.

The practices described below are encouraged in apprentices by expert mathematical thinkers. Students who engage in these practices, individually and with their classmates, discover ideas and gain insights that spur them to pursue mathematics beyond the classroom walls. They learn that effort counts in mathematical achievement. Encouraging these practices in students of all ages should be as much a goal of the mathematics curriculum as the learning of specific content.

1 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

2 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

3 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

4 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving

problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

5 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

6 Reason abstractly and quantitatively.

Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

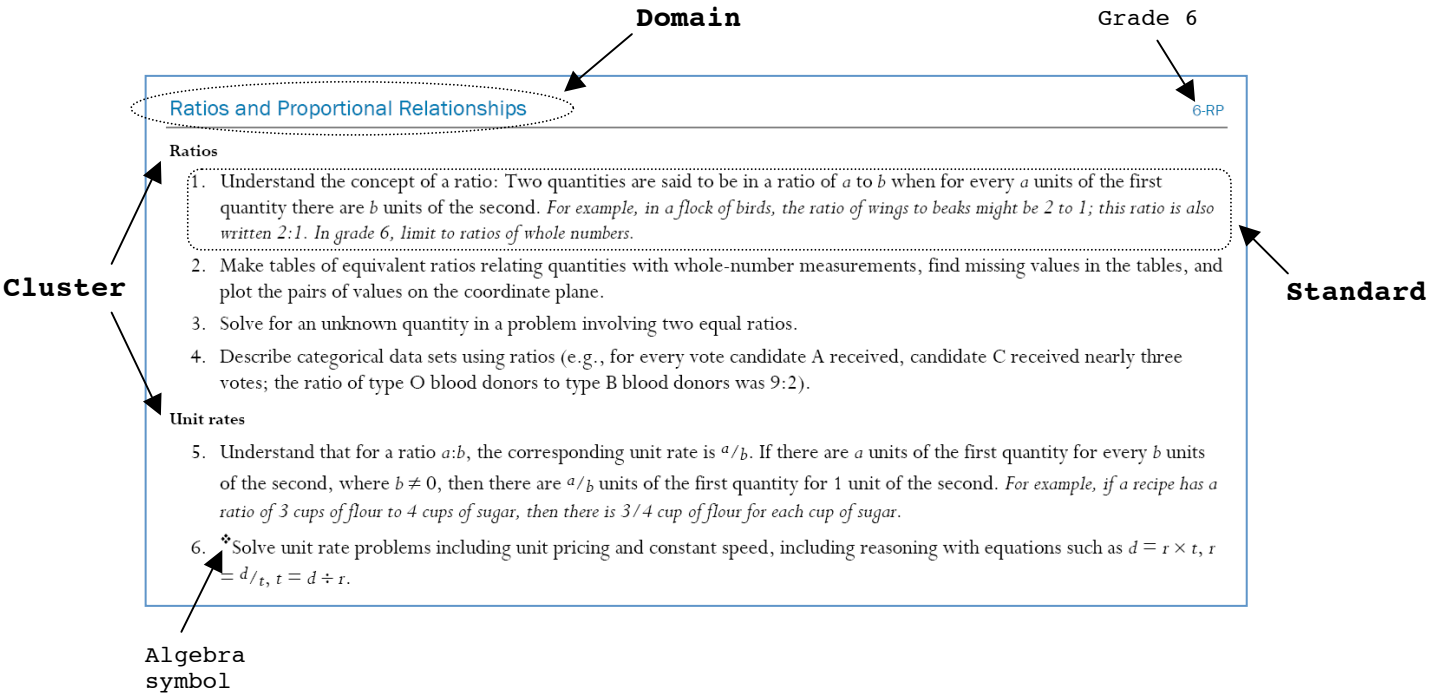
7 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, 2-by-2 tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

8 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include as pencil and paper, concrete models, ruler, protractor, calculator, spreadsheet, computer algebra system, statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from and the limitations and those of their output. For example, mathematically proficient high school students interpret graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

How to read the grade level standards



Standards define what students should understand and be able to do. **Clusters** are groups of related standards. Note that standards from different clusters may sometimes be closely related, because mathematics is a connected subject. **Domains** are larger groups of related standards. For each grade level in Grades K–8, the standards are organized into four or five domains. Standards from different domains may sometimes be closely related.

Algebra Symbol: Key standards for the development of algebraic thinking in Grades K–5 are indicated by ❖.

Dotted Underlines: Dotted underlines, for example, decade words, indicate terms that are explained in the Glossary. In each grade, underlining is used for the first occurrence of a defined term, but not in subsequent occurrences.

Note on Grade Placement of Topics. What students can learn at any particular grade level depends upon what they have learned before. Ideally then, each standard in this document might have been phrased in the form, "Students who already know A should next come to learn B." But in the year 2010 this approach is unrealistic—not least because existing education research cannot specify all such learning pathways. Of necessity therefore, grade placements for specific topics have been made on the basis of state and international comparisons and the collective experience and collective professional judgment of educators, researchers and mathematicians. One promise of common state standards is that over time they will allow research on learning progressions to inform and improve the design of standards to a much greater extent than is possible today. Learning opportunities will continue to vary across schools and school systems, and educators should make every effort to meet the needs of individual students based on their current understanding.

Note on Ordering of Topics within a Grade. These standards do not dictate curriculum. In particular, just because topic A appears before topic B in the standards for a given grade, it does not necessarily mean that topic A must be taught before topic B. A teacher might prefer to teach topic B before topic A, or might choose to highlight connections by teaching topic A and topic B at the same time. Or, a teacher might prefer to teach a topic of his or her own choosing that leads, as a byproduct, to students reaching the standards for topics A and B.

Overview of the Mathematics Standards
Grades K–5

This table shows the domains and clusters in each grade K–5

	K	*1*	*2*	*3*	*4*	*5*
Number—Counting and Cardinality	• Number names • Counting to tell the number of objects • Comparing and ordering numbers					
Number—Operations and the Problems They Solve	• Composing and decomposing numbers; addition and subtraction	• Addition and subtraction • Describing situations and solving problems with addition and subtraction	• Addition and subtraction • Describing situations and solving problems with addition and subtraction	• Multiplication and division • Describing situations and solving problems with multiplication and division	• Multiplication and Division • Problem solving with the four operations	
Number—Base Ten	• Two-digit numbers • Composing and decomposing ten	• Numbers up to 100 • Adding and subtracting in base ten	• Numbers up to 1000 • Adding and subtracting in base ten	• Numbers up to 10,000 • Adding and subtracting in base ten • Multiplying and dividing in base ten	• Numbers up to 100,000 • Multiplying and dividing in base ten	• Dividing whole numbers in base ten • Decimal concepts • Operations on decimals
Number—Fractions				• Fractions as representations of numbers • Fractional quantities	• Operations on fractions • Decimal concepts	• Fraction equivalence • Operations on fractions
Measurement and Data	• Direct measurement • Representing and interpreting data	• Length measurement • Time measurement • Representing and interpreting data	• Length measurement • Time and money • Representing and interpreting data	• The number line and units of measure • Perimeter and area • Representing and interpreting data	• The number line and units of measure • Perimeter and area • Angle measurement • Representing and interpreting data	• Units of measure • Volume • Representing and interpreting data
Geometry	• Shapes, their attributes, and spatial reasoning	• Shapes, their attributes, and spatial reasoning	• Shapes, their attributes, and spatial reasoning	• Properties of 2-dimensional shapes • Structuring rectangular shapes	• Lines and angles • Line symmetry	• Coordinates • Plane figures

Overview of the Mathematics Standards
Grades 6–8

This table shows the domains and clusters in each grade 6–8.

	Grade 6	Grade 7	Grade 8
Ratios and Proportional Relationships	• Ratios • Unit rates	• Analyzing proportional relationships • Percent	
The Number System	• Operations on fractions • The system of rational numbers	• The system of rational numbers • The system of real numbers	• The system of real numbers
Expressions and Equations	• Expressions • Quantitative relationships and the algebraic approach to problems	• Expressions • Quantitative relationships and the algebraic approach to solving problems	• Slopes of lines in the coordinate plane • Linear equations and systems
Functions			• Function concepts • Functional relationships between quantities
Geometry	• Properties of area, surface area, and volume	• Congruence and similarity • Angles	• Congruence and similarity • The Pythagorean Theorem • Plane and solid geometry
Statistics and Probability	• Variability and measures of center • Summarizing and describing distributions	• Situations involving randomness • Random sampling to draw inferences about a population • Comparative inferences about two populations	• Patterns of association in bivariate data

Mathematics | Kindergarten

In Kindergarten, instructional time should focus on two critical areas: (1) representing, comparing and ordering whole numbers and joining and separating sets; (2) describing shapes and space. More learning time in Kindergarten should be devoted to number than to other topics.

(1) Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; creating a set with a given number of objects; comparing and ordering sets or numerals; and modeling simple joining and separating situations with objects. They choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

(2) Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic shapes, such as squares, triangles, circles, rectangles, (regular) hexagons, and (isosceles) trapezoids, presented in a variety of ways (e.g., with different sizes or orientations), as well as three-dimensional shapes such as spheres, cubes, and cylinders. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

Number—Counting and Cardinality K-NCC

Number names

1. Say the number name sequence to 100.
2. Know the decade words to ninety and recite them in order ("ten, twenty, thirty, …").
3. Say the number name sequence forward or backward beginning from a given number within the known sequence (instead of always beginning at 1).
4. Write numbers from 1 to 20 in base-ten notation.

Counting to tell the number of objects

5. Count to answer "how many?" questions about as many as 20 things. *Objects may be arranged in a line, a rectangular array, a circle, or a scattered configuration.*
6. Understand that when counting objects,
   a. The number names are said in the standard order.
   b. Each object is paired with one and only one number name.
   c. The last number name said tells the number of objects counted.
7. Understand that when counting forward, each successive number name refers to a quantity that is 1 larger.

Comparing and ordering numbers

8. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies. *Include groups with up to ten objects.*
9. Compare and put in order numbers between 1 and 10 presented in written symbols: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Number—Operations and the Problems They Solve K-NOP

Composing and decomposing numbers; addition and subtraction

1. Understand addition as putting together—e.g., finding the number of objects in a group formed by putting two groups together. Understand subtraction as taking apart—e.g., finding the number of objects left when a one group is taken from another.
2. Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations. *Note that drawings need not show details, but should show the mathematics in the problem. (This note also applies wherever drawings are mentioned in subsequent standards.)*
3. ❖Decompose numbers less than or equal to 10 into pairs in various ways, e.g., using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3). Compose numbers whose sum is less than or equal to 10, e.g., using objects or drawings, and record each composition by a drawing or equation (e.g., 3 + 1 = 4).❖
4. Compose and decompose numbers less than or equal to 10 in two different ways, and record compositions and decompositions by drawings or equations. *For example, 7 might be composed or decomposed in two different ways by a drawing showing how a group of 2 and a group of 5 together make the same number as do a group of 3 and a group of 4.*
5. ❖Understand that addition and subtraction are related. *For example, when a group of 9 is decomposed into a group of 6 and a group of 3, this means not only 9 = 6 + 3 but also 9 – 3 = 6 and 9 – 6 = 3.*
6. ❖Solve addition and subtraction word problems, and calculate additions and subtractions within 10, e.g., using objects or drawings to represent the problem.
7. Fluently add and subtract, for sums and minuends of 5 or less.

Number—Base Ten K-NBT

Two-digit numbers

1. Understand that 10 can be thought of as a bundle of ones—a unit called a "ten."
2. Understand that a teen number is composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
3. Compose and decompose teen numbers into a ten and some ones, e.g., by using objects or drawings, and record the compositions and decompositions in base-ten notation. *For example, 10 + 8 = 18 and 14 = 10 + 4.*
4. Put in order numbers presented in base-ten notation from 1 to 20 (inclusive), and be able to explain the reasoning.
5. Understand that a decade word refers to one, two, three, four, five, six, seven, eight, or nine tens.

6. Understand that the two digits of a two-digit number represent amounts of tens and ones. *In 29, for example, the 2 represents two tens and the 9 represents nine ones.*

Composing and decomposing ten

7. Decompose 10 into pairs of numbers, e.g., by using objects or drawings, and record each decomposition with a drawing or equation.
8. Compose numbers to make 10, e.g., by using objects or drawings, and record each composition with a drawing or equation.
9. ❖For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

Measurement and Data K-MD

Direct measurement

1. Understand that objects have measurable attributes, such as length or weight. A single object might have several measurable attributes of interest.
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of" the attribute. *For example, directly compare the heights of two books and identify which book is taller.*

Representing and interpreting data

3. Classify objects or people into given categories; count the numbers in each category and sort the categories by count. *Limit category counts to be less than or equal to 10.*

Geometry K-G

Shapes, their attributes, and spatial reasoning

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.
2. Understand that names of shapes apply regardless of the orientation or overall size of the shape. *For example, a square in any orientation is still a square. Students may initially need to physically rotate a shape until it is "level" before they can correctly name it.*
3. Understand that shapes can be two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").
4. Understand that shapes can be seen as having parts, such as sides and vertices ("corners"), and that shapes can be put together to compose other shapes.
5. Analyze and compare a variety of two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, component parts (e.g., number of sides and vertices) and other attributes (e.g., having sides of equal length).
6. Combine two- or three-dimensional shapes to solve problems such as deciding which puzzle piece will fit into a place in a puzzle.

Mathematics | Grade 1

In Grade 1, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for additions and subtractions within 20; (2) developing understanding of whole number relationships, including grouping in tens and ones, (3) developing understanding of linear measurement and measuring lengths, and (4) composing and decomposing geometric shapes.

(1) Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model "put together/take apart," "add to," "take from," and "compare" situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (i.e., adding two is the same as counting on two). They use properties of addition (commutativity and associativity) to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the inverse relationship between addition and subtraction.

(2) Students compare and order whole numbers (at least to 100), to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). They understand the sequential order of the counting numbers and their relative magnitudes through activities such as representing numbers on paths of numbered things.

(3) Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as partitioning (the mental activity of decomposing the length of an object into equal-sized units) and transitivity (e.g., in terms of length, if object A is longer than object B and object B is longer than object C, then object A is longer than object C). They understand linear measure as an iteration of units, and use rulers and other measurement tools with that understanding.

(4) Students compose and decompose plane and solid figures (e.g., put two congruent isosceles triangles together to make a rhombus), building understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine solid and plane figures, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

Number—Operations and the Problems They Solve 1-NOP

Addition and subtraction

1. ❖Understand the properties of addition.
   a. Addition is commutative. *For example, if 3 cups are added to a stack of 8 cups, then the total number of cups is the same as when 8 cups are added to a stack of 3 cups; that is, 8 + 3 = 3 + 8.*
   b. Addition is associative. *For example, 4 + 3 + 2 can be found by first adding 4 + 3 = 7 then adding 7 + 2 = 9, or by first adding 3 + 2 = 5 then adding 4 + 5 = 9.*
   c. 0 is the additive identity.
2. ❖Explain and justify properties of addition and subtraction, e.g., by using representations such as objects, drawings, and story contexts. Explain what happens when:
   a. The order of addends in a sum is changed in a sum with two addends.
   b. 0 is added to a number.
   c. A number is subtracted from itself.
   d. One addend in a sum is increased by 1 and the other addend is decreased by 1. *Limit to two addends.*
3. ❖Understand that addition and subtraction have an inverse relationship. *For example, if 8 + 2 = 10 is known, then 10 – 2 = 8 and 10 – 8 = 2 are also known.*
4. ❖Understand that when all but one of three numbers in an addition or subtraction equation are known, the unknown number can be found. *Limit to cases where the unknown number is a whole number.*
5. Understand that addition can be recorded by an expression (e.g., 6 + 3), or by an equation that shows the sum (e.g., 6 + 3 = 9). Likewise, subtraction can be recorded by an expression (e.g., 9 – 5), or by an equation that shows the difference (e.g., 9 – 5 = 4).

Describing situations and solving problems with addition and subtraction

6. Understand that addition and subtraction apply to situations of adding-to, taking-from, putting together, taking apart, and comparing. *See Glossary, Table 1.*
7. ❖Solve word problems involving addition and subtraction within 20, e.g., by using objects, drawings and equations to represent the problem. *Students should work with all of the addition and subtraction situations shown in the Glossary, Table 1, solving problems with unknowns in all positions, and representing these situations with equations that use a symbol for the unknown (e.g., a question mark or a small square). Grade 1 students need not master the more difficult problem types.*
8. Solve word problems involving addition of three whole numbers whose sum is less than or equal to 20.

Number—Base Ten 1-NBT

Numbers up to 100

1. Read and write numbers to 100.
2. Starting at any number, count to 100 or beyond.
3. Understand that when comparing two-digit numbers, if one number has more tens, it is greater; if the amount of tens is the same in each number, then the number with more ones is greater.
4. Compare and order two-digit numbers based on meanings of the tens and ones digits, using > and < symbols to record the results of comparisons.

Adding and subtracting in base ten

5. Calculate mentally, additions and subtractions within 20.
   a. Use strategies that include counting on; making ten (for example, 7 + 6 = 7 + 3 + 3 = 10 + 3 = 13); and decomposing a number (for example, 17 – 9 = 17 – 7 – 2 = 10 – 2 = 8).
6. Demonstrate fluency in addition and subtraction within 10.
7. Understand that in adding or subtracting two-digit numbers, one adds or subtracts like units (tens and tens, ones and ones) and sometimes it is necessary to compose or decompose a higher value unit.
8. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count.
9. Add one-digit numbers to two-digit numbers, and add multiples of 10 to one-digit and two-digit numbers.
10. Explain addition of two-digit numbers using concrete models or drawings to show composition of a ten or a hundred.

11. ❖Add two-digit numbers to two-digit numbers using strategies based on place value, properties of operations, and/or the inverse relationship between addition and subtraction; explain the reasoning used.

Measurement and Data 1-MD

Length measurement

1. Order three objects by length; compare the length of two objects indirectly by using a third object.
2. Understand that the length of an object can be expressed numerically by using another object as a length unit (such as a paper-clip, yardstick, or inch length on a ruler). The object to be measured is partitioned into as many equal parts as possible with the same length as the length unit. The length measurement of the object is the number of length units that span it with no gaps or overlaps. *For example, "I can put four paperclips end to end along the pencil, so the pencil is four paperclips long."*
3. Measure the length of an object by using another object as a length unit.

Time measurement

4. Tell time from analog clocks in hours and half- or quarter-hours.

Representing and interpreting data

5. Organize, represent, and interpret data with several categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Geometry 1-G

Shapes, their attributes, and spatial reasoning

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size) for a wide variety of shapes.
2. Understand that shapes can be joined together (composed) to form a larger shape or taken apart (decomposed) into a collection of smaller shapes. Composing multiple copies of some shapes creates tilings. *In this grade, "circles," "rectangles," and other shapes include their interiors as well as their boundaries.*
3. Compose two-dimensional shapes to create a unit, using cutouts of rectangles, squares, triangles, half-circles, and quarter-circles. Form new shapes by repeating the unit.
4. Compose three-dimensional shapes to create a unit, using concrete models of cubes, right rectangular prisms, right circular cones, and right circular cylinders. Form new shapes by repeating the unit. *Students do not need to learn formal names such as "right rectangular prism."*
5. Decompose circles and rectangles into two and four equal parts. Describe the parts using the words *halves*, *fourths*, and *quarters*, and using the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the parts. Understand that decomposing into more equal shares creates smaller shares.
6. Decompose two-dimensional shapes into rectangles, squares, triangles, half-circles, and quarter-circles, including decompositions into equal shares.

Mathematics | Grade 2

In Grade 2, instructional time should focus on three critical areas: (1) developing understanding of base-ten notation; (2) developing fluency with additions and subtractions within 20 and fluency with multi-digit addition and subtraction; and (3) describing and analyzing shapes.

(1) Students develop an understanding of the base-ten system (at least to 1000). Their understanding of the base-ten system includes ideas of counting in units (twos, fives, and tens) and multiples of hundreds, tens, and ones, as well as number relationships, including comparing and ordering. They understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).

(2) Students use their understanding of addition to develop fluency with additions and subtractions within 20. They solve arithmetic problems by applying their understanding of models for addition and subtraction (such as combining or separating sets or using number lines that begin with zero), relationships and properties of numbers, and properties of addition. They develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of two-digit whole numbers. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences. They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers; understand and explain why the procedures work based on their understanding of base-ten notation and properties of operations; and use them to solve problems.

(3) Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding attributes of two- and three-dimensional space such as area and volume, and properties such as congruence and symmetry that they will learn about in later grades.

Number—Operations and the Problems They Solve 2-NOP

Addition and subtraction

1. ❖Explain and justify properties of addition and subtraction, e.g., by using representations such as objects, drawings, and story contexts. Include properties such as:
   a. Changing the order of addends does not change their sum.
   b. Subtracting one addend from a sum of two numbers results in the other addend.
   c. If more is subtracted from a number, the difference is decreased, and if less is subtracted the difference is increased.
   d. In an addition equation, each addend can be decomposed and the parts can be recombined in any order without changing the sum. *For example, $5 + 3 = 8$. Because 5 decomposes as $4 + 1$, the first addend can be replaced by $4 + 1$, yielding $(4 + 1) + 3 = 8$. Recombining in two different orders: $4 + 4 = 8$, also $7 + 1 = 8$.*

Describing situations and solving problems with addition and subtraction

2. ❖Solve word problems involving addition and subtraction within 100, e.g., by using drawings or equations to represent the problem. *Students should work with all of the addition and subtraction situations shown in the Glossary, Table 1, solving problems with unknown sums, addends, differences, minuends, and subtrahends, and representing these situations with equations that use a symbol for the unknown (e.g., a question mark or a small square). Focus on the more difficult problem types.*
3. Solve two-step word problems involving addition and subtraction within 100, e.g., by using drawings or equations to represent the problem.

Number—Base Ten 2-NBT

Numbers up to 1000

1. Understand that 100 can be thought of as a bundle of tens—a unit called a "hundred."
2. Read and write numbers to 1000 using base-ten notation, number names, and expanded form.
3. Count within 1000; skip count by 2s, 5s, 10s, and 100s.
4. Understand that when comparing three-digit numbers, if one number has more hundreds, it is greater; if the amount of hundreds is the same in each number, then the number with more tens is greater. If the amount of tens and hundreds is the same in each number, then the number with more ones is greater.
5. Compare and order three-digit numbers based on meanings of the hundreds, tens, and ones digits.

Adding and subtracting in base ten

6. Fluently add and subtract within 20. By end of Grade 2, know from memory sums of one-digit numbers.
7. Mentally compute sums and differences of multiples of 10. *For example, mentally calculate $130 - 80$.*
8. Understand that in adding or subtracting three-digit numbers, one adds or subtracts like units (hundreds and hundreds, tens and tens, ones and ones) and sometimes it is necessary to compose or decompose a higher value unit.
9. Given a number from 100 to 900, mentally find 10 more or 10 less than the number, and mentally find 100 more or 100 less than the number, without counting.
10. Understand that algorithms are predefined steps that give the correct result in every case, while strategies are purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. *For example, one might mentally compute $503 - 398$ as follows: $398 + 2 = 400$, $400 + 100 = 500$, $500 + 3 = 503$, so the answer is $2 + 100 + 3$, or 105.*
11. ❖ Compute sums and differences of one-, two-, and three-digit numbers using strategies based on place value, properties of operations, and/or the inverse relationship between addition and subtraction; explain the reasoning used.
12. ❖Explain why addition and subtraction strategies and algorithms work, using place value and the properties of operations. *Include explanations supported by drawings or objects. A range of reasonably efficient algorithms may be covered, not only the standard algorithm.*
13. Compute sums of two three-digit numbers, and compute sums of three or four two-digit numbers, using the standard algorithm; compute differences of two three-digit numbers using the standard algorithm.

Measurement and Data 2-MD

Length measurement

1. Understand that 1 inch, 1 foot, 1 centimeter, and 1 meter are conventionally defined lengths used as standard units.

2. Measure lengths using measurement tools such as rulers, yardsticks and measuring tapes; understand that these tools are used to find out how many standard length units span an object with no gaps or overlaps, when the 0 mark of the tool is aligned with an end of the object.
3. Understand that when measuring a length, if a smaller unit is used, more copies of that unit are needed to measure the length than would be necessary if a larger unit were used.
4. Understand that units can be decomposed into smaller units, e.g., 1 foot can be decomposed into 12 inches and 1 meter can be decomposed into 100 centimeters. A small number of long units might compose a greater length than a large number of small units.
5. Understand that lengths can be compared by placing objects side by side, with one end lined up. The difference in lengths is how far the longer extends beyond the end of the shorter.
6. Understand that a sum of two whole numbers can represent a combination of two lengths; a difference of two whole numbers can represent a difference in length; find total lengths and differences in lengths using addition and subtraction.

Time and money

7. Find time intervals between hours in one day.
8. Solve word problems involving dollar bills, quarters, dimes, nickels and pennies. *Do not include dollars and cents in the same problem.*

Representing and interpreting data

9. Generate measurement data by measuring whole-unit lengths of several objects, or by making repeated measurements of the same object. Show the measurements by making a dot plot, where the horizontal scale is marked off in whole-number units.
10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with several categories. Connect representations on bar graph scales, rulers, and number lines that begin with zero. Solve simple Put Together/Take Apart and Compare problems using information presented in a bar graph. *See Glossary, Table 1.*

Geometry 2-G

Shapes, their attributes, and spatial reasoning

1. Understand that different categories of shapes (e.g., rhombuses, trapezoids, rectangles, and others) can be united into a larger category (e.g., quadrilaterals) on the basis of shared attributes (e.g., having four straight sides).
2. Identify and name polygons of up to six sides by the number of their sides or angles.
3. Recognize rectangles, rhombuses, squares and trapezoids as examples of quadrilaterals; draw examples of quadrilaterals that do not belong to any of these subcategories.
4. Draw and identify shapes that have specific attributes, such as number of equal sides or number of equal angles. *Sizes of lengths and angles are compared directly or visually, not compared by measuring.*
5. Recognize objects as resembling spheres, right circular cylinders, and right rectangular prisms. *Students do not need to learn formal names such as "right rectangular prism."*
6. Decompose circular and rectangular objects into two, three, or four equal parts. Describe the parts using the words *halves*, *thirds*, *half of*, *a third of*, etc.; describe the wholes as two halves, three thirds, four fourths. Recognize that a half, a third, or a fourth of a circular or rectangular object—a graham cracker, for example—is the same size regardless of its shape.

Mathematics | Grade 3

In Grade 3, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division and strategies for multiplication and division within 100; (2) developing understanding of fractions, starting with unit fractions; (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes. Multiplication, division, and fractions are the most important developments in Grade 3.

(1) Students develop an understanding of the meanings of multiplication and division of whole numbers through the use of representations such as equal-sized groups, arrays, area models, and equal jumps on number lines for multiplication; and successive subtraction, partitioning, and sharing for division. Through this process, numbers themselves take on new meaning and are no longer only counters for single objects. They represent groups, a number of groups (for example, 3 teams of 6 people), or a comparative factor (3 times as long).

Students use properties of operations to calculate products of whole numbers. They use increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the inverse relationship between multiplication and division.

(2) Students develop an understanding of a definition of a fraction, beginning with unit fractions. They use fractions to represent parts of a whole or distances on a number line that begins with zero. Students understand that the size of a fractional part is relative to the size of the whole (for example, $\frac{1}{4}$ of a mile is longer than $\frac{3}{4}$ of a foot, even though $\frac{1}{4} < \frac{3}{4}$), and they are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing and ordering fractions using by models or strategies based on noticing common numerators or denominators.

(3) Students recognize area as an attribute of two-dimensional regions. They understand that area can be quantified by finding the total number of same-size units of area required to cover the shape without gaps or overlaps. They understand that a 1-unit by 1-unit square is the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area measure to the area model used to represent multiplication, and they use this connection to justify using multiplication to determine the area of a rectangle. Students contrast area with perimeter.

(4) Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify the shapes by their sides and angles, and connect these with definitions of shapes. Students investigate, describe, and reason about decomposing and combining polygons to make other polygons. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of attributes and properties of two-dimensional objects.

Number—Operations and the Problems They Solve 3-NOP

Multiplication and division

1. Understand that multiplication of whole numbers is repeated addition. *For example, 5 × 7 means 7 added to itself 5 times. Products can be represented by rectangular arrays, with one factor the number of rows and the other the number of columns.*
2. ❖Understand the properties of multiplication.
   a. Multiplication is commutative. *For example, the total number in 3 groups with 6 things each is the same as the total number in 6 groups with 3 things each, that is, 3 × 6 = 6 × 3.*
   b. Multiplication is associative. *For example, 4 × 3 × 2 can be calculated by first calculating 4 × 3 = 12 then calculating 12 × 2 = 24, or by first calculating 3 × 2 = 6 then calculating 4 × 6 = 24.*
   c. 1 is the multiplicative identity.
   d. Multiplication distributes over addition (the distributive property). *For example, 5 × (3 + 4) = (5 × 3) + (5 × 4).*
3. ❖Explain and justify properties of multiplication and division, e.g., by using representations such as objects, drawings, and story contexts. Include properties such as:
   a. Changing the order of two factors does not change their product.
   b. The product of a number and 1 is the number.
   c. Dividing a nonzero number by itself yields 1.
   d. Multiplying a quantity by a nonzero number, then dividing by the same number, yields the original quantity.
   e. When one factor in a product is multiplied by a number and another factor divided by the same number, the product is unchanged. *Limit to multiplying and dividing by numbers that result in whole-number quotients.*
   f. Products where one factor is a one-digit number can be computed by decomposing one factor as the sum of two numbers, multiplying each number by the other factor, and adding the two products.
4. ❖Understand that multiplication and division have an inverse relationship. *For example, if 5 × 7 = 35 is known, then 35 ÷ 5 = 7 and 35 ÷ 7 = 5 are also known. The division 35 ÷ 5 means the number which yields 35 when multiplied by 5; because 5 × 7 = 35, then 35 ÷ 5 = 7.*
5. ❖Understand that when all but one of three numbers in a multiplication or division equation are known, the unknown number can be found. *Limit to cases where the unknown number is a whole number.*

Describing situations and solving problems with multiplication and division

6. Understand that multiplication and division apply to situations with equal groups, arrays or area, and comparing. *See Glossary, Table 2.*
7. ❖Solve word problems involving multiplication and division within 100, using an equation with a symbol for the unknown to represent the problem. *This standard is limited to problems with whole-number quantities and whole-number quotients. Focus on situations described in the Glossary, Table 2.*
8. ❖Solve one- or two-step word problems involving the four operations. *This standard is limited to problems with whole-number quantities and whole-number quotients.*
9. Understand that multiplication and division can be used to compare quantities (see Glossary, Table 2); solve multiplicative comparison problems with whole numbers (problems involving the notion of "times as much").

Number—Base Ten 3-NBT

Numbers up to 10,000

1. Understand that 1000 can be thought of as a bundle of hundreds—a unit called a "thousand."
2. Read and write numbers to 10,000 using base-ten notation, number names, and expanded form.
3. Count within 10,000; skip count by 10s, 100s and 1000s.
4. Understand that when comparing four-digit numbers, if one number has more thousands, it is greater; if the amount of thousands is the same in each number, then the number with more hundreds is greater; and so on. Compare and order four-digit numbers based on meanings of the digits.

Adding and subtracting in base ten

5. Mentally calculate sums and differences of multiples of 10, 100, and 1000. *For example, mentally calculate 1300 – 800*
6. Given a number from 1000 to 9000, mentally find 100 more or 100 less than the number, and mentally find 1000 more or 1000 less than the number, without counting.

Multiplying and dividing in base ten

7. ❖Understand that the distributive property is at the heart of strategies and algorithms for multiplication and division computations with numbers in base-ten notation; use the distributive property and other properties of operations to explain patterns in the multiplication table and to derive new multiplication and division equations from known ones. *For example, the distributive property makes it possible to multiply 4 × 7 by decomposing 7 as 5 + 2 and using 4 × 7 = 4 × (5 + 2) = (4 × 5) + (4 × 2) = 20 + 8 = 28.*
8. Fluently multiply one-digit numbers by 10.
9. Use a variety of strategies for multiplication and division within 100. By end of Grade 3, know from memory products of one-digit numbers where one of the factors is 2, 3, 4, or 5.

Number—Fractions 3-NF

Fractions as representations of numbers

1. Understand that a unit fraction corresponds to a point on a number line. *For example, 1/3 represents the point obtained by decomposing the interval from 0 to 1 into three equal parts and taking the right-hand endpoint of the first part. In Grade 3, all number lines begin with zero.*
2. Understand that fractions are built from unit fractions. *For example, 5/4 represents the point on a number line obtained by marking off five lengths of ¼ to the right of 0.*
3. Understand that two fractions are equivalent (represent the same number) when both fractions correspond to the same point on a number line. Recognize and generate equivalent fractions with denominators 2, 3, 4, and 6 (e.g., $1/2 = 2/4$, $4/6 = 2/3$), and explain the reasoning.
4. Understand that whole numbers can be expressed as fractions. *Three important cases are illustrated by the examples 1 = 4/4, 6 = 6/1, and 7 = (4 × 7)/4. Expressing whole numbers as fractions can be useful for solving problems or making calculations.*

Fractional quantities

5. Understand that fractions apply to situations where a whole is decomposed into equal parts; use fractions to describe parts of wholes. *For example, to show 1/3 of a length, decompose the length into 3 equal parts and show one of the parts.*
6. Compare and order fractional quantities with equal numerators or equal denominators, using the fractions themselves, tape diagrams, number line representations, and area models. Use > and < symbols to record the results of comparisons.

Measurement and Data 3-MD

The number line and units of measure

1. Understand that a number line has an origin (0) and a unit (1), with whole numbers one unit distance apart. Use number lines to represent problems involving distances, elapsed time, amounts of money and other quantities. *In such problems, the interval from 0 to 1 may represent a unit of distance, time, money, etc.*
2. Understand that a unit of measure can be decomposed into equal-sized parts, whose sizes can be represented as fractions of the unit. Convert measurements in one unit to measurements in a smaller or a larger unit, and solve problems involving such mixed units (e.g., feet and inches, weeks and days).

Perimeter and area

3. Understand and use concepts of area measurement.
   a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
   b. A plane figure which can be covered without gaps or overlaps by *n* unit squares has an area of *n* square units. Areas of some other figures can be measured by using fractions of unit squares or using figures whose areas have been found by decomposing other figures.
   c. When measuring an area, if a smaller unit of measurement is used, more units must be iterated to measure the area in those units.
   d. Determine and compare areas by counting square units. *Use cm^2, m^2, in^2, ft^2, and improvised units.*
4. Understand that multiplication of whole numbers can be represented by area models; a rectangular region that is *a* length units by *b* length units (where *a* and *b* are whole numbers) and tiled with unit squares illustrates why the rectangle encloses an area of *a* × *b* square units.
5. Solve problems involving perimeters of polygons.
   a. Add given side lengths, and multiply for the case of equal side lengths.

b. ❖Find an unknown length of a side in a polygon given the perimeter and all other side lengths; represent these problems with equations involving a letter for the unknown quantity.

c. Exhibit rectangles with the same perimeter and different area, and with the same area and different perimeter.

Representing and interpreting data

6. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *Include single-unit scales and multiple-unit scales; for example, each square in the bar graph might represent 1 pet, 5 pets, or 10 pets.*

7. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a dot plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

Geometry 3-G

Properties of 2-dimensional shapes

1. Understand that a given category of plane figures (e.g., triangles) has subcategories (e.g., isosceles triangles) defined by special properties.

2. Describe, analyze, compare and classify two-dimensional shapes by their properties and connect these properties to the classification of shapes into categories and subcategories (e.g., squares are "special rectangles" as well as "special rhombuses"). *Focus on triangles and quadrilaterals.*

Structuring rectangular shapes

3. Understand that rectangular regions can be tiled with squares in rows and columns, or decomposed into such arrays.

4. Structure a rectangular region spatially by decomposing it into rows and columns of squares. Determine the number of squares in the region using that spatial structure (e.g., by multiplication or skip counting).

5. Understand that shapes can be decomposed into parts with equal areas; the area of each part is a unit fraction of the whole. *For example, when a shape is partitioned into 4 parts with equal area, the area of each part is ¼ of the area of the shape.*

Mathematics | Grade 4

In Grade 4, instructional time should focus on four critical areas: (1) continuing to develop understanding and fluency with whole number multiplication, and developing understanding of multi-digit whole number division; (2) developing an understanding of addition and subtraction of fractions with like denominators, multiplication of fractions by whole numbers, and division of whole numbers with fractional answers; (3) developing an understanding of area; and (4) understanding that geometric figures can be analyzed and classified using properties such as having parallel sides, perpendicular sides, particular angle measures, and symmetry.

(1) Students use understandings of multiplication to develop fluency with multiplication and division within 100. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models, equal intervals on a number line), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate products or mentally calculate products. They develop fluency with efficient procedures, including the standard algorithm, for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate quotients and mentally calculate quotients, depending upon the context and the numbers involved.

(2) Students develop understanding of operations with fractions. They apply their understandings of fractions as built from unit fractions, and use fraction models to represent the addition and subtraction of fractions with like denominators. Students use the meaning of fractions and the meaning of multiplication to understand and explain why the procedure for multiplying a fraction by a whole number makes sense. They understand and explain the connection between division and fractions.

(3) Students develop their understanding of area. They understand and apply the area formula for rectangles and also find areas of shapes that can be decomposed into rectangles. They select appropriate units, strategies (e.g., decomposing shapes), and tools for solving problems that involve estimating and measuring area.

(4) Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

Number—Operations and the Problems They Solve 4-NOP

Multiplication and division

1. Find the factor pairs for a given whole number less than or equal to 100; recognize prime numbers as numbers greater than 1 with exactly one factor pair. *Example: The factor pairs of 42 are {42, 1}, {21, 2}, {14, 3}, {7, 6}.*

Problem solving with the four operations

2. ❖Solve multistep word problems involving the four operations with whole numbers.
3. ❖Solve problems posed with both whole numbers and fractions. Understand that while quantities in a problem might be described with whole numbers, fractions, or decimals, the operations used to solve the problem depend on the relationships between the quantities regardless of which number representations are involved.
4. Assess the reasonableness of answers using mental computation and estimation strategies including rounding to the nearest 10 or 100.

Number—Base Ten 4-NBT

Numbers up to 100,000

1. Understand that a digit in one place represents ten times what it represents in the place to its right. *For example, 7 in the thousands place represents 10 times as many as than 7 in the hundreds place.*
2. Read, write and compare numbers to 100,000 using base-ten notation, number names, and expanded form.

Multiplying and dividing in base ten

3. Understand how the distributive property and the expanded form of a multi-digit number can be used to calculate products of multi-digit numbers.
   a. ❖The product of a one-digit number times a multi-digit number is the sum of the products of the one-digit number with the summands in the expanded form of the multi-digit number. Illustrate this numerically and visually using equations, rectangular arrays, area models, and tape diagrams.
   b. Algorithms for multi-digit multiplication can be derived and explained by writing multi-digit numbers in expanded form and applying the distributive property.
4. Fluently multiply and divide within 100. By end of Grade 4, know from memory products of one-digit numbers where one of the factors is 6, 7, 8, or 9.
5. Mentally calculate products of one-digit numbers and one-digit multiples of 10, 100, and 1000 (e.g., 7×6000). Mentally calculate whole number quotients with divisors of 10 and 100.
6. Compute products and whole number quotients of two-, three- or four-digit numbers and one-digit numbers, and compute products of two two-digit numbers, using strategies based on place value, the properties of operations, and/or the inverse relationship between multiplication and division; explain the reasoning used.
7. Explain why multiplication and division strategies and algorithms work, using place value and the properties of operations. *Include explanations supported by drawings, equations, or both. A range of reasonably efficient algorithms may be covered, not only the standard algorithms.*
8. Compute products of two-digit numbers using the standard algorithm, and check the result using estimation.
9. Given two whole numbers, find an equation displaying the largest multiple of one which is less than or equal to the other. *For example, given 325 and 7, the equation $325 = 46 \times 7 + 3$ shows the largest multiple of 7 less than or equal to 325.*

Number—Fractions 4-NF

Operations on fractions

1. Understand addition of fractions:
   a. Adding or subtracting fractions with the same denominator means adding or subtracting copies of unit fractions. *For example, $2/3 + 4/3$ is 2 copies of $1/3$ plus 4 copies of $1/3$, or 6 copies of $1/3$ in all, that is $6/3$.*
   b. Sums of related fractions can be computed by replacing one with an equivalent fraction that has the same denominator as the other. *For example, the sum of the related fractions $2/3$ and $1/6$ can be computed by rewriting $2/3$ as $4/6$ and computing $4/6 + 1/6 = 5/6$.*
2. Compute sums and differences of fractions with like denominators, add and subtract related fractions within 1 (e.g., $1/2 + 1/4$, $3/10 + 4/100$, $7/8 - 1/4$), and solve word problems involving these operations.

3. ❖ Understand that the meaning of multiplying a fraction by a whole number comes from interpreting multiplication by a whole number as repeated addition. *For example, $3 \times 2/5 = 6/5$ because $3 \times 2/5 = 2/5 + 2/5 + 2/5 = 6/5$.*
4. Solve word problems that involve multiplication of fractions by whole numbers; represent multiplication of fractions by whole numbers using tape diagrams and area models that explain numerical results.
5. ❖Understand that fractions give meaning to the quotient of any whole number by any non-zero whole number. *For example, $3 \div 4 = 3/4$, because $3/4$ multiplied by 4 equals 3. (The division $3 \div 4$ means the number which yields 3 when multiplied by 4.)*
6. Solve word problems that involve non-whole number quotients of whole numbers; represent quotients of whole numbers using tape diagrams and area models that explain numerical results.

Decimal concepts

7. Understand that a two-digit decimal is a sum of fractions with denominators 10 and 100. *For example, 0.34 is $3/10 + 4/100$.*
8. Use decimals to hundredths to describe parts of wholes; compare and order decimals to hundredths based on meanings of the digits; and write fractions of the form $a/10$ or $a/100$ in decimal notation. *Use $>$ and $<$ symbols to record the results of comparisons.*

Measurement and Data 4-MD

The number line and units of measure

1. Understand that the unit length on a number line (interval from 0 to 1) can be divided into parts of equal fractional length. Draw number line representations of problem situations involving length, height, and distance including fractional or decimal units. *For example, show distances along a race course to tenths of a mile on a number line, by dividing the unit length into 10 equal parts to get parts of length $1/10$; the endpoint of the segment of $1/10$ length from 0 represents $1/10$ of a mile from the starting point of the race. In Grade 4, all numbers lines begin with zero.*

Perimeter and area

2. Understand that if a region is decomposed into several disjoint pieces, then the area of the region can be found by adding the areas of the pieces (when these areas are expressed in the same units).
3. ❖Apply the formulas for area of squares and rectangles. Measure and compute whole-square-unit areas of objects and regions enclosed by geometric figures which can be decomposed into rectangles. *Limit to situations requiring products of one-or two-digit numbers.*
4. ❖Find one dimension of a rectangle, given the other dimension and the area or perimeter; find the length of one side of a square, given the area or perimeter. Represent these problems using equations involving a letter for the unknown quantity.

Angle measurement

5. Understand what an angle is and how it is measured:
   a. An angle is formed by two rays with a common endpoint.
   b. An angle is measured by reference to a circle with its center at the common endpoint of the rays. The measure of an angle is based on the fraction of the circle between the points where the two rays intersect the circle.
   c. A one-degree angle turns through $1/360$ of a circle, where the circle is centered at the common endpoint of its rays; the measure of a given angle is the number of one-degree angles turned with no gaps or overlaps.
6. Measure angles in whole-number degrees using a protractor; sketch angles of specified measure; ❖find the measure of a missing part of an angle, given the measure of the angle and the measure of a part of it, representing these problems with equations involving a letter for the unknown quantity.

Representing and interpreting data

7. Make a dot plot to display a data set of measurements in fractions of a unit ($1/2$, $1/4$, $1/8$). Solve problems involving addition and subtraction of fractions by using information presented in dot plots. *For example, from a dot plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

Geometry 4-G

Lines and angles

1. Draw points, lines, line segments, rays, angles, and perpendicular and parallel lines; identify these in plane figures.
2. Identify right angles, and angles smaller than or greater than a right angle in geometric figures; recognize right triangles.

3. Classify shapes based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of specified size.

Line symmetry

4. Understand that a line of symmetry for a geometric figure is a line across the figure such that the figure can be folded along the line into matching parts
5. Identify line-symmetric figures; given a horizontal or vertical line and a drawing that is not a closed figure, complete the drawing to create a figure that is symmetric with respect to the given line.

Mathematics | Grade 5

In Grade 5, instructional time should focus on four critical areas: (1) developing fluency with addition and subtraction of fractions, developing understanding of the multiplication of fractions and of division of fractions in limited cases (fractions divided by whole numbers and whole numbers divided by unit fractions); (2) developing understanding of and fluency with division of multi-digit whole numbers; (3) developing understanding of and fluency with addition, subtraction, multiplication, and division of decimals; and (4) developing understanding of volume.

(1) Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They develop fluency in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the inverse relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing fractions by whole numbers and whole numbers by unit fractions.)

(2) Students develop fluency with division of whole numbers; understand why procedures work based on the meaning of base-ten notation and properties of operations; and use these procedures to solve problems. Based on the context of a problem situation, they select the most useful form of the quotient for the answer and interpret it appropriately.

(3) Students apply their understandings of models for decimals, decimal notation, and properties of operations to compute sums and differences of finite decimals. They develop fluency in these computations, and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of finite decimals efficiently and accurately.

(4) Students recognize volume as an attribute of three-dimensional space. They understand that volume can be quantified by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve problems.

Number—Base Ten 5-NBT

Dividing whole numbers in base ten

1. Compute quotients of two-, three-, and four-digit whole numbers and two-digit whole numbers using strategies based on place value, the properties of operations, and/or the inverse relationship between multiplication and division; explain the reasoning used.
2. Explain why division strategies and algorithms work, using place value and the properties of operations. *Include explanations supported by drawings, equations, or both. A range of reasonably efficient algorithms may be covered, not only the standard algorithm.*
3. Use the standard algorithm to compute quotients of two-, three- and four-digit whole numbers and two-digit whole numbers, expressing the results as an equation (e.g., 145 = 11 × 13 + 2 or 120 ÷ 7 = 17 1/7).

Decimal concepts

4. Read, write, and compare numbers expressed as decimals. Understand that a digit in one place represents ten times what it represents in the place to its right. *For example, 7 in the hundredths place represents 10 times as many as 7 in the thousandths place.*
5. Round decimals (to hundredths) to the nearest whole number.
6. Write fractions in decimal notation for fractions with denominators 2, 4, 5, 8, 10, and 100.

Operations on decimals

7. Understand that in adding or subtracting finite decimals, one adds or subtracts like units (tenths and tenths, hundredths and hundredths, etc.) and sometimes it is necessary to compose or decompose a higher value unit.
8. Fluently find 0.1 more than a number and less than a number; 0.01 more than a number and less than a number; and 0.001 more than a number and less than a number, for numbers expressed as finite decimals.
9. Compute sums and differences of finite decimals by expressing the decimals as fractions and adding the fractions. *For example, 0.05 + 0.91 = 5/100 + 91/100 = 96/100 or 0.96.*
10. Compute sums, differences, products, and quotients of finite decimals using strategies based on place value, the properties of operations, and/or the inverse relationships between addition and subtraction and between multiplication and division; explain the reasoning used. *For example, transform 1.5 ÷ 0.3 into 15 ÷ 3 = 5.*
11. Explain why strategies and algorithms for computations with finite decimals work. *Include explanations supported by drawings, equations, or both. A range of reasonably efficient algorithms may be covered, not only the standard algorithm.*
12. Use the standard algorithm for each of the four operations on decimals (to hundredths).
13. Solve word problems involving operations on decimals.

Number—Fractions 5-NF

Fraction equivalence

1. ❖Understand fraction equivalence:
   a. Multiplying the numerator and denominator of a fraction by the same nonzero whole number produces an equivalent fraction. *For example, 2/3 = (2 × 4)/(3 × 4) = 8/12. (1/3 is 4 copies of 1/12, so 2/3 is 8 copies of 1/12.)*
   b. Equivalent fractions correspond to the same point on a number line. *In Grade 5, all numbers lines begin with zero.*
   c. When the numerators of equivalent fractions are divided by their denominators, the resulting quotients are the same.
2. Identify pairs of equivalent fractions; given two fractions with unlike denominators, find two fractions with the same denominator and equivalent to each.
3. Compare and order fractions with like or unlike denominators, e.g., by finding equivalent fractions with the same denominator, and describe the sizes of fractional quantities from a context with reference to the context. *Compare using the fractions themselves, tape diagrams or number line representations, and area models.*

Operations on fractions

4. Understand that sums and differences of fractions with unlike denominators can be computed by replacing each with an equivalent fraction so that the resulting fractions have the same denominator. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12.*
5. Compute sums and differences of fractions with like or unlike denominators, and solve word problems involving addition and subtraction of fractions. Estimate fraction sums and differences to assess the reasonableness of results.

6. ❖ Understand that multiplying a fraction by a/b means taking a parts of a decomposition of the fraction into b equal parts. *For example, to multiply $2/3 \times 4/5 = 8/15$, one may decompose a whole of size $4/5$ into 3 equal parts; each part has size $4/15$. Two of these parts then make $8/15$, so $2/3 \times 4/5 = 8/15$. (In general, $a/b \times p/q = ap/bq$.) This standard includes multiplication of a whole number by a fraction, by writing the whole number as fraction with denominator 1.*
7. Understand that the area of a rectangle with side lengths a/b and c/d is the product $a/b \times p/q$. This extends the area formula for rectangles to fractional side lengths, and also allows products of fractions to be represented visually as areas of rectangles.
8. ❖Explain and justify the properties of operations with fractions, e.g., by using equations, number line representations, area models, and story contexts.
9. Understand division of unit fractions by whole numbers and division of whole numbers by unit fractions:
   a. Dividing a unit fraction 1/b by a whole number a results in a smaller unit fraction 1/a × b. *For example, $1/3 \div 2 = 1/6$ because when $1/3$ is divided into 2 equal parts, the size of each part is $1/6$; a third of a pound of cheese shared between two people will give each person a sixth of a pound. (Using the inverse relationship between multiplication and division: $1/3 \div 2 = 1/6$ because $1/6 \times 2 = 1/3$.)*
   b. Dividing a whole number a by a unit fraction 1/b results in a greater whole number a × b. *For example, $2 \div 1/3 = 6$ because 6 is the number of $1/3$s in 2; two pounds of cheese will make six portions of a third of a pound each. (Using the inverse relationship between multiplication and division: $2 \div 1/3 = 6$ because $6 \times 1/3 = 2$.)*
10. Calculate products of fractions, and quotients of unit fractions and nonzero whole numbers (with either as divisor), and solve word problems involving these operations. Represent these operations using equations, area models and length models.
11. Understand that a mixed number such as 3 2/5 represents the sum of a whole number and a fraction less than one. Because a whole number can be represented as a fraction (3 = 3/1), and the sum of two fractions is also a fraction, a mixed number also represents a fraction (3 2/5 = 3 + 2/5 = 15/5 + 2/5 = 17/5). Write fractions as equivalent mixed numbers and vice versa.

Measurement and Data 5-MD

Units of measure

1. Understand that quantities expressed in like units can be added or subtracted giving a sum or difference with the same unit; different quantities may be multiplied to obtain a new kind of quantity (e.g., as when two lengths are multiplied to compute an area, or when an area and a length are multiplied to compute a volume).
2. Understand that when measuring a quantity, if a smaller unit is used, more units must be iterated to measure the quantity in those units.
3. Convert among different-sized standard measurement units within a given measurement system (e.g., feet to yards, centimeters to meters) and use conversion in solving multi-step word problems.

Volume

4. Understand concepts of volume measurement:
   a. A cube with side length 1 unit (a unit cube) is said to have "one cubic unit" of volume, and can be used to measure volume.
   b. The volume of a right rectangular prism with whole-unit side lengths can be found by packing it with unit cubes and using multiplication to count their number. *For example, decomposing a right rectangular prism 3 length units wide by 5 units deep by 2 units tall shows that its volume is $3 \times 5 \times 2$ cubic units. The base of the prism has area 3×5 square units, so the volume can also be expressed as the height times the area of the base.*
   c. When measuring a volume, if a smaller unit is used, more units must be iterated to measure the volume in those units.
   d. If a solid figure is decomposed into several disjoint pieces, then the volume enclosed by the figure can be found by adding the volumes of the pieces (when these volumes are expressed in the same units).
5. Decompose right rectangular prisms into layers of arrays of cubes; determine and compare volumes of right rectangular prisms, and objects well described as right rectangular prisms, by counting cubic units (using cm^3, m^3, in^3, ft^3, and improvised units).

Representing and interpreting data

6. Make a dot plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in dot plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

Coordinates

1. Understand that a pair of perpendicular number lines, called axes, defines a coordinate system.
   a. Their intersection is called the origin, usually arranged to coincide with the 0 on each line.
   b. A given point in the plane can be located by using an ordered pair of numbers, called its coordinates. The first number indicates how far to travel from the origin in the direction of one axis, the second number indicates how far to travel in the direction of the second axis.
   c. To avoid ambiguity, conventions dictate that the names of the two axes and the coordinates correspond (e.g., *x*-axis and *x*-coordinate, *y*-axis and *y*-coordinate).
2. Graph points in the first quadrant of the coordinate plane, and identify the coordinates of graphed points. Where ordered pairs arise in a problem situation, interpret the coordinate values in the context of the situation.

Plane figures

3. Understand that properties belonging to a category of plane figures also belong to all subcategories of that category. *For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*
4. Classify plane figures in a hierarchy based on properties.

Mathematics | Grade 6

In Grade 6, instructional time should focus on four critical areas: (1) connecting ratio and rate to whole number multiplication and division; (2) developing understanding of and fluency with division of fractions and developing fluency with multiplication of fractions; (3) developing understanding of and using formulas to determine areas of two-dimensional shapes and distinguishing between volume and surface area of three-dimensional shapes; and (4) writing, interpreting, and using expressions and equations.

(1) Students use reasoning about multiplication and division with quantities to solve ratio and rate problems. By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students extend whole number multiplication and division to ratios and rates. Thus students expand their repertoires of problems in which multiplication and division can be used to solve problems, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.

(2) Students use the meaning of fractions, the meanings of multiplication and division, and the inverse relationship between multiplication and division to understand and explain why the procedures for dividing fractions make sense. Students are able to add, subtract, multiply, and divide fractions fluently, and use these operations to solve problems, including multi-step problems and problems involving measurement.

(3) Students reason about relationships among shapes to determine area and surface area. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposition into pieces whose area they can determine.

(4) Students write mathematical expressions and equations that correspond to given situations, they evaluate expressions, and they use expressions and formulas to solve problems. Students understand that a variable is a letter standing for a number, where the number is unknown, or where, for the purpose at hand, it can be any number in the domain of interest. Students understand that expressions in different forms can be equivalent, and they use the laws of arithmetic to rewrite expressions to represent a total quantity in a different way (such as to represent it more compactly or to feature different information). Students know that the solutions of an equation are the values of the variables that make the equation true. Students use properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. Students construct and analyze tables, such as tables of quantities that are in equivalent ratios, and they use equations (such as $3x = y$) to describe relationships in a table.

Having represented and analyzed data in Grades K–5, students in Grade 6 begin a serious engagement with statistics. The study of variability in data distinguishes statistics from mathematics. Students beginning their study of variability must first recognize statistical questions as those that anticipate variability in the answers. From this conceptual beginning, they learn to describe and summarize distributions of data—an activity that goes beyond merely computing summary statistics to include assessing the shape of a distribution and considering other issues as described in the standards.

Ratios and Proportional Relationships 6-RP

Ratios

1. Understand the concept of a ratio: Two quantities are said to be in a ratio of a to b when for every a units of the first quantity there are b units of the second. *For example, in a flock of birds, the ratio of wings to beaks might be 2 to 1; this ratio is also written 2:1. In Grade 6, limit to ratios of whole numbers.*
2. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane.
3. Solve for an unknown quantity in a problem involving two equal ratios.
4. Describe categorical data sets using ratios (e.g., for every vote candidate A received, candidate C received nearly three votes; the ratio of type O blood donors to type B blood donors was 9:2).

Unit rates

5. Understand that for a ratio a:b, the corresponding unit rate is a/b. If there are a units of the first quantity for every b units of the second, where $b \neq 0$, then there are a/b units of the first quantity for 1 unit of the second. *For example, if a recipe has a ratio of 3 cups of flour to 4 cups of sugar, then there is 3/4 cup of flour for each cup of sugar.*
6. ❖Solve unit rate problems including unit pricing and constant speed, including reasoning with equations such as $d = r \times t$, $r = d/t$, $t = d \div r$.

The Number System 6-NS

Operations on fractions

1. Understand that the properties of operations apply to, and can be used with, addition and multiplication of fractions.
2. Understand that division of fractions is defined by viewing a quotient as the solution for an unknown-factor multiplication problem. *For example, (2/3) ÷ (5/7) = 14/15 because (5/7) × (14/15) = (2/3).*
3. Solve word problems requiring arithmetic with fractions, using the properties of operations and converting between forms as appropriate; estimate to check reasonableness of answers.

The system of rational numbers

4. Understand that a number is a point on the number line.
5. Understand that some quantities have opposite directions, such as elevation above and below sea level or money received and spent. These quantities can be described using positive and negative numbers.
6. Understand that number lines familiar from previous grades can be extended to represent negative numbers to the left of zero. *Number lines can also be vertically oriented, as when a coordinate system is formed. Then the conventional terms "to the right of 0" and "to the left of 0" conventionally become "above 0" and "below 0."*
   a. Two different numbers, such as 7 and –7, that are equidistant from zero on a number line are said to be opposites of one another. The opposite of the opposite of a number is the number itself, e.g., –(–3) = 3. The opposite of 0 is 0.
   b. The absolute value of a number q, written $|q|$, is its distance from zero, and is always positive or zero.
   c. Fractions and their opposites form a system of numbers called the rational numbers, represented by points on a number line. Whole numbers and their opposites form the integers, which are contained in the rational numbers.
   d. Previous ways of comparing positive numbers can be extended to the rational numbers. The statement $p > q$ means that p is located to the right of q on a number line, while $p < q$ means that p is located to the left of q on a number line. Comparisons can also be made by reasoning appropriately about signed quantities (e.g, –3 > –7 makes sense because –3°C is a higher temperature than –7°C). The way two numbers compare does not always agree with the way their absolute values compare; for example, –3 > –7, but $|-3| < |-7|$.
7. Find and position rational numbers, including integers, on a number line.
8. Use rational numbers to describe quantities such as elevation, temperature, account balance and so on. Compare these quantities, recording the results of comparisons using > and < symbols.
9. Graph points and identify coordinates of points on the coordinate plane in all four quadrants. Where ordered pairs arise in a problem situation, interpret the coordinate values in the context of the situation.

Expressions and Equations 6-EE

Expressions

1. Understand that an expression records operations with numbers or with letters standing for numbers. *For example, the expression $2 \cdot (8 + 7)$ records adding 8 and 7 then multiplying by 2; the expression $5 - y$ records subtracting y from 5. Focus on the operations of addition, subtraction, multiplication and division, with some attention to square or cube roots.*
2. Understand the use of variables in expressions and algebraic conventions:
   a. A letter is used to stand for a number in an expression in cases where the number is unknown, or where, for the purpose at hand, it can be any number in a domain of interest. Such a letter is called a variable.
   b. If a variable appears in an expression more than once (e.g., as in $t + 3t$), that variable is understood to refer to the same number in each instance.
   c. The multiplication symbol can be omitted when writing products of two or more variables or of a number and a variable. *For example, the expressions xy and $2a$ indicate $x \times y$ and $2 \times a$, respectively.*
3. Describe the structure and elements of simple expressions using correct terminology (sum, term, product, factor, quotient, coefficient); describe an expression by viewing one or more of its parts as a single entity. *For example, describe the expression $2 \cdot (8 + 7)$ as a product of two factors, by viewing $(8 + 7)$ as a single entity. The second factor is itself a sum of two terms.*
4. Understand and generate equivalent expressions:
   a. Understand that two expressions are equivalent if they name the same number regardless of which numbers the variables in them stand for. *For example, the expressions $x + 3$ and $4x$ are not equivalent, even though they happen to name the same number in the case when x stands for 1.*
   b. Understand that applying the laws of arithmetic to an expression results in an equivalent expression. *For example, applying the distributive law to the expression $3 \cdot (2 + x)$ leads to the equivalent expression $6 + 3x$. Applying the distributive law to $y + y + y$ leads to the equivalent expression $y \times (1 + 1 + 1)$, i.e., $y \times 3$ and then the commutative law of multiplication leads to the equivalent expression $3y$.*
   c. Generate equivalent expressions to reinterpret the meaning of an expression. *For example, $2t + 3t$ records the addition of twice a quantity to three times itself; applying the distributive law leads to the equivalent expression $5t$, so that the original expression can be reinterpreted as recording five times the quantity.*

Quantitative relationships and the algebraic approach to problems

5. Understand that an equation is a statement that two expressions are equal, and a solution to an equation is a replacement value of the variable (or replacement values for all the variables if there is more than one) that makes the equation true.
6. Using the idea of maintaining equality between both sides of the equation, solve equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.
7. Choose variables to represent quantities in a word problem, and construct simple expressions or equations to solve the problem by reasoning about the quantities.
8. Understand that a variable can be used to represent a quantity that can change, often in relationship to another changing quantity, and an equation can express one quantity, thought of as the dependent variable, in terms of other quantities, thought of as the independent variables; represent a relationship between two quantities using equations, graphs, and tables; translate between any two of these representations. *For example, describe the terms in a sequence $t = 3, 6, 9, 12, \ldots$ of multiples of 3 by writing the equation $t = 3n$ for $n = 1, 2, 3, 4, \ldots$.*

Geometry 6-G

Properties of area, surface area, and volume

1. Understand that plane figures can be decomposed, reassembled, and completed into new figures; use this technique to derive area formulas.
2. Find the areas enclosed by right triangles, other triangles, special quadrilaterals, and polygons (by composing into rectangles or decomposing into triangles and other shapes).
3. Understand that three-dimensional figures can be formed by joining rectangles and triangles along their edges to enclose a solid region with no gaps or overlaps. The surface area is the sum of the areas of the enclosing rectangles and triangles.
4. Find the surface area of cubes, prisms and pyramids (include the use of nets to represent these figures).
5. Solve problems involving area, volume and surface area of objects.
6. Give examples of right rectangular prisms with the same surface area and different volumes, and with the same volume and different surface areas.

7. ❖Use exponents and symbols for square roots and cube roots to express the area of a square and volume of a cube in terms of their side lengths, and to express their side lengths in terms of their area or volume.

Statistics and Probability 6-SP

Variability and measures of center

1. Understand that a statistical question is one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question , but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.*
2. Understand that a set of data generated by answers to a statistical question typically shows variability—not all of the values are the same—and yet often the values show an overall pattern, often with a tendency to cluster.
   a. A measure of center for a numerical data set summarizes all of its values using a single number. The median is a measure of center in the sense that approximately half the data values are less than the median, while approximately half are greater. The mean is a measure of center in the sense that it is the value that each data point would take on if the total of the data values were redistributed fairly, and in the sense that it is the balance point of a data distribution shown on a dot plot.
   b. A measure of variation for a numerical data set describes how its values vary using a single number. The interquartile range and the mean absolute deviation are both measures of variation.

Summarizing and describing distributions

3. Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
4. Summarize numerical data sets, such as by:
   a. Reporting the number of observations.
   b. Describing the nature of the variable, including how it was measured and its units of measurement. *Data sets can include fractional values at this grade but not negative values.*
   c. Describing center and variation, as well as describing any overall pattern and any striking deviations from the overall pattern.
5. Relate the choice of the median or mean as a measure of center to the shape of the data distribution being described and the context in which it is being used. Do the same for the choice of interquartile range or mean average deviation as a measure of variation. *For example, why are housing prices often summarized by reporting the median selling price, while students' assigned grades are often based on mean homework scores?*

Mathematics | Grade 7

In Grade 7, instructional time should focus on four critical areas: (1) developing understanding of and applying proportional relationships; (2) developing understanding of operations with rational numbers and solving linear equations; (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence; and (4) drawing inferences about populations based on samples.

(1) Students extend their understanding of ratios and develop understanding of proportionality to solve single- and multi-step problems. Students use their understanding of ratios and proportionality to solve a wide variety of percent problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease. Students solve problems about similar objects (including geometric figures) by using scale factors that relate corresponding lengths between the objects or by using the fact that relationships of lengths within an object are preserved in similar objects. Students graph proportional relationships and understand the unit rate informally as a measure of the steepness of the related line, called the slope. They distinguish proportional relationships from other relationships.

(2) Students develop a unified understanding of number, recognizing fractions, decimals, and percents as different representations of rational numbers. Students extend addition, subtraction, multiplication, and division and their properties to all rational numbers, including integers and numbers represented by complex fractions and negative fractions. By applying the laws of arithmetic, and by viewing negative numbers in terms of everyday contexts (e.g., amounts owed or temperatures below zero), students explain why the rules for adding, subtracting, multiplying, and dividing with negative numbers make sense. They use the arithmetic of rational numbers as they formulate and solve linear equations in one variable and use these equations to solve problems.

(3) Students use ideas about distance and angles, how they behave under dilations, translations, rotations and reflections, and ideas about congruence and similarity to describe and analyze figures and situations in two- and three-dimensional space and to solve problems, including multi-step problems. Students prove that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students apply this reasoning about similar triangles to solve problems, such as finding heights and distances. Students see the plausibility of the formulas for the circumference and area of a circle. For example, in the case of area, they may do so by reasoning about how lengths and areas scale in similar figures or by decomposing a circle or circular region and rearranging the pieces.

(4) Students build on their previous work with single data distributions to compare two data distributions and address questions about differences between populations. They begin informal work with random sampling to generate data sets and learn about the importance of representative samples for drawing inferences.

Ratios and Proportional Relationships 7-RP

Analyzing proportional relationships

1. Form ratios of nonnegative rational numbers and compute corresponding unit rates. *For example, a person might walk ½ mile in each ¼ hour; the unit rate for this ratio is (1/2)/(1/4) miles per hour, equivalently 2 miles per hour. Include ratios of lengths, areas and other quantities, including when quantities being compared are measured in different units.*
2. Recognize situations in which two quantities covary and have a constant ratio. (The quantities are then said to be in a proportional relationship and the unit rate is called the constant of proportionality.) Decide whether two quantities that covary are in a proportional relationship, e.g., by testing for equivalent ratios or graphing on a coordinate plane.
3. Compute unit rates and solve proportional relationship problems in everyday contexts, such as shopping, cooking, carpentry, party planning, etc. Represent proportional relationships by equations that express how the quantities are related via the constant of proportionality or unit rate. *For example, total cost, t, is proportional to the number, n, purchased at a constant price, p; this relationship can be expressed as $t = pn$.*
4. Plot proportional relationships on a coordinate plane where each axis represents one of the two quantities involved, observe that the graph is a straight line through the origin, and find unit rates from a graph. Explain what a point (x, y) means in terms of the situation, with special attention to the points (0, 0) and $(1, r)$ where r is the unit rate.
5. Compare tables, graphs, formulas, diagrams, and verbal descriptions that represent or partially represent proportional relationships; explain correspondences among the representations including how the unit rate is shown in each.

Percent

6. Understand that percentages are rates per 100. For example, 30% of a quantity means $30/100$ times the quantity. A percentage can be a complex fraction, as in $3.75\% = 3.75/100$.
7. Find a percentage of a quantity; solve problems involving finding the whole given a part and the percentage.
8. Solve multistep percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error, expressing monthly rent as a percentage of take-home pay.*

The Number System 7-NS

The system of rational numbers

1. Understand that the rules for manipulating fractions extend to complex fractions.
2. Understand and perform addition and subtraction with rational numbers:
   a. Understand that on a number line, the sum $p + q$ is the number located a distance $|q|$ from p, to the right of p if q is positive and to the left of p if q is negative. A number and its opposite are additive inverses (i.e., their sum is zero).
   b. Compute sums of signed numbers using the laws of arithmetic. *For example, $7 + (-3) = 4$ because $7 + (-3) = (4 + 3) + (-3) = 4 + [3 + (-3)] = 4 + [0] = 4$.*
   c. Understand that subtraction of rational numbers is defined by viewing a difference as the solution of an unknown-addend addition problem. Subtraction of a rational number gives the same answer as adding its additive inverse.
   d. Explain and justify rules for adding and subtracting rational numbers, using a number line and practical contexts. *For example, relate $r + (-s) = r - s$ to a bank transaction; explain why $p - (q + r) = p - q - r$.*
   e. Understand that the additive inverse of a sum is the sum of the additive inverses, that is $-(p + q) = -p + -q$. *For example, $-(6 + -2) = (-6) + 2$ because $[6 + (-2)] + [(-6) + 2] = [6 + (-6)] + [(-2) + 2] = [0] + [0] = 0$.*
3. Understand and perform multiplication and division with rational numbers:
   a. Understand that the extension of multiplication from fractions to rational numbers is determined by the requirement that multiplication and addition satisfy the laws of arithmetic, particularly the distributive law, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers.
   b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p/q is a rational number, then $-(p/q) = (-p)/q = p/(-q)$.
   c. Calculate products and quotients of rational numbers, and use multiplication and division to solve word problems. *Include signed quantities.*

The system of real numbers

4. Understand that there are numbers that are not rational numbers, called irrational numbers, e.g., π and $\sqrt{2}$. Together the rational and irrational numbers form the real number system. In school mathematics, the real numbers are assumed to satisfy the laws of arithmetic.

Expressions and Equations 7-EE

Expressions

1. Interpret numerical expressions at a level necessary to calculate their value using a calculator or spreadsheet. For expressions with variables, use and interpret conventions of algebraic notation, such as $y/2$ is $y \div 2$ or $1/2 \times y$; $(3 \pm y)/5$ is $(3 \pm y) \div 5$ or $1/5 \times (3 \pm y)$; a^2 is $a \times a$, a^3 is $a \times a \times a$, a^2b is $a \times a \times b$.
2. Generate equivalent expressions from a given expression using the laws of arithmetic and conventions of algebraic notation. Include:
   a. Adding and subtracting linear expressions, as in $(2x + 3) + x + (2 - x) = 2x + 5$.
   b. Factoring, as in $4x + 4y = 4(x + y)$ or $5x + 7x + 10y + 14y = 12x + 24y = 12(x + 2y)$.
   c. Simplifying, as in $-2(3x - 5) + 4x = 10 - 2x$ or $x/3 + (x - 2)/4 = 7x/12 - 1/2$.

Quantitative relationships and the algebraic approach to problems

3. Choose variables to represent quantities in a word problem, and construct simple equations to solve the problem by reasoning about the quantities.
   a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are nonnegative rational numbers and the solution is a nonnegative rational number. Fluently solve equations of these forms, e.g., by undoing the operations involved in producing the expression on the left.
   b. Solve the same word problem arithmetically and algebraically. *For example, "J. has 4 packages of balloons and 5 single balloons. In all, he has 21 balloons. How many balloons are in a package?" Solve this problem arithmetically (using a sequence of operations on the given numbers), and also solve it by using a variable to stand for the number of balloons in a package, constructing an equation such as $4b + 5 = 21$ to describe the situation then solving the equation.*
   c. Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, $P + 0.05P = 1.05P$ means that "increase by 5%" is the same as "multiply by 1.05."*

Geometry 7-G

Congruence and similarity

1. Verify experimentally the fact that a rigid motion (a sequence of rotations, reflections, and translations) preserves distance and angle, e.g., by using physical models, transparencies, or dynamic geometry software:
   a. Lines are taken to lines, and line segments to line segments of the same length.
   b. Angles are taken to angles of the same measure.
   c. Parallel lines are taken to parallel lines.
2. Understand the meaning of congruence: a plane figure is congruent to another if the second can be obtained from the first by a rigid motion.
3. Verify experimentally that a dilation with scale factor k preserves lines and angle measure, but takes a line segment of length L to a line segment of length kL.
4. Understand the meaning of similarity: a plane figure is similar to another if the second can be obtained from the first by a similarity transformation (a rigid motion followed by a dilation).
5. Solve problems involving similar figures and scale drawings. *Include computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.*
6. Use informal arguments involving approximation by lines, squares, and cubes to see that a similarity transformation with a scale factor of k leaves angle measures unchanged, changes lengths by a factor of k, changes areas by a factor of k^2, and changes volumes by a factor of k^3.
7. Know the formulas relating the area, radius and circumference of a circle and solve problems requiring the use of these formulas; give an informal derivation of the relationship between the circumference and area of a circle.

Angles

8. Justify facts about the angle sum of triangles, exterior angles, and alternate interior angles created when parallel lines are cut by a transversal, e.g., by using physical models, transparencies, or dynamic geometry software to make rigid motions and give informal arguments. *For example, arrange three copies of the same triangle so that the three angles appear to form a line, and give an argument in terms of transversals why this is so.*
9. Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

Situations involving randomness

1. Simulate situations involving randomness using random numbers generated by a calculator or a spreadsheet or taken from a table. *For example, if you guess at all ten true/false questions on a quiz, how likely are you to get at least seven answers correct?*
2. Use proportional reasoning to predict relative frequencies of outcomes for situations involving randomness, but for which a theoretical answer can be determined. For *example, when rolling a number cube 600 times, one would predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times. How far off might your prediction be? Use technology to generate multiple samples to approximate a distribution of sample proportions. Repeat the process for smaller sample sizes.*

Random sampling to draw inferences about a population

3. Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.
4. Understand the importance of measures of variation in sample quantities (like means or proportions) in reasoning about how well a sample quantity estimates or predicts the corresponding population quantity.
5. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*

Comparative inferences about two populations

6. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean average deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*
7. Use measures of center and measures of variability for numerical data from uniform random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade book are generally longer than the words in a chapter of a sixth-grade book.*

Mathematics | Grade 8

In Grade 8, instructional time should focus on three critical areas: (1) solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; (3) understanding and applying the Pythagorean Theorem.

(1) Students use linear equations, and systems of linear equations to represent, analyze, and solve a variety of problems. Students recognize proportions ($y/x = m$ or $y = mx$) as a special case of linear equations, $y = mx + b$, understanding that the constant of proportionality (m) is the slope and the graphs are lines through the origin. They understand that the slope (m) of a line is a constant rate of change, so that if the input or x-coordinate changes by an amount A, the output or y-coordinate changes by the amount mA. Students also formulate and solve linear equations in one variable and use these equations to solve problems. Students also use a linear equation to describe the association between two quantities in a data set (such as arm span vs. height for students in a classroom). At this grade, fitting the model, and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question.

Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students solve systems of two linear equations in two variables and relate the systems to pairs of lines in the plane; these intersect, are parallel, or are the same line. Students use linear equations, systems of linear equations, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

(2) Students grasp the concept of a function as a rule that assigns to each element of its domain exactly one element of its range. They use function notation and understand that functions describe situations where one quantity determines another. They can translate among verbal, tabular, graphical, and algebraic representations of functions (noting that tabular and graphical representations are usually only partial representations), and they describe how aspects of the function are reflected in the different representations.

(3) Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem is valid, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons.

The Number System 8-NS

The system of real numbers

1. Understand informally that every number on a number line has a decimal expansion, which can be found for rational numbers using long division. Rational numbers are those with repeating decimal expansions (this includes finite decimals which have an expansion that ends in a sequence of zeros).
2. Informally explain why $\sqrt{2}$ is irrational.
3. Use rational approximations (including those obtained from truncating decimal expansions) to compare the size of irrational numbers, locate them approximately on a number line, and estimate the value of expressions (e.g., π^2). *For example, show that the square root of 2 is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*

Expressions and Equations 8-EE

Linear equations in one variable

1. Understand that a linear equation in one variable might have one solution, infinitely many solutions, or no solutions. Which of these possibilities is the case can be determined by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).
2. Solve linear equations with rational number coefficients, including equations that require expanding expressions using the distributive law and collecting like terms.

Linear equations in two variables

3. Understand that the slope of a non-vertical line in the coordinate plane has the same value for any two distinct points used to compute it. This can be seen using similar triangles.
4. Understand that two lines with well-defined slopes are parallel if and only if their slopes are equal.
5. Understand that the graph of a linear equation in two variables is a line, the set of pairs of numbers satisfying the equation. If the equation is in the form $y = mx + b$, the graph can be obtained by shifting the graph of $y = mx$ by b units (upwards if b is positive, downwards if b is negative). The slope of the line is m.
6. Understand that a proportional relationship between two variable quantities y and x can be represented by the equation $y = mx$. The constant m is the unit rate, and tells how much of y per unit of x.
7. Graph proportional relationships and relationships defined by a linear equation; find the slope and interpret the slope in context.
8. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.*

Systems of linear equations

9. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
10. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because the quantity $3x + 2y$ cannot simultaneously be 5 and 6.*
11. Solve and explain word problems leading to two linear equations in two variables.
12. Solve problems involving lines and their equations. *For example, decide whether a point with given coordinates lies on the line with a given equation; construct an equation for a line given two points on the line or one point and the slope; given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

Functions 8-F

Function concepts

1. Understand that a function from one set (called the domain) to another set (called the range) is a rule that assigns to each element of the domain (an input) exactly one element of the range (the corresponding output). The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. *Function notation is not required in Grade 8.*
2. Evaluate expressions that define functions, and solve equations to find the input(s) that correspond to a given output.
3. Compare properties of two functions represented in different ways (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

4. Understand that a function is linear if it can be expressed in the form $y = mx + b$ or if its graph is a straight line. *For example, the function $y = x^2$ is not a linear function because its graph contains the points (1,1), (–1,1) and (0,0), which are not on a straight line.*

Functional relationships between quantities

5. Understand that functions can describe situations where one quantity determines another.
6. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship; from two (x, y) values, including reading these from a table; or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
7. Describe qualitatively the functional relationship between two quantities by reading a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Geometry 8-G

Congruence and similarity

1. Use coordinate grids to transform figures and to predict the effect of dilations, translations, rotations and reflections.
2. Explain using rigid motions the meaning of congruence for triangles as the equality of all pair of sides and all pairs of angles.
3. Give an informal explanation using rigid motions of the SAS and ASA criteria for triangle congruence, and use them to prove simple theorems.
4. Explain using similarity transformations the meaning of similarity for triangles as the equality of all pairs of angles and the proportionality of all pairs of sides.
5. Give an informal explanation using similarity transformations of the AA and SAS criteria for triangle similarity, and use them to prove simple theorems.

The Pythagorean Theorem

6. The side lengths of a right triangle are related by the Pythagorean Theorem. Conversely, if the side lengths of a triangle satisfy the Pythagorean Theorem, it is a right triangle.
7. Explain a proof of the Pythagorean Theorem and its converse.
8. Use the Pythagorean Theorem to determine unknown side lengths in right triangles and to solve problems in two and three dimensions.
9. Use the Pythagorean Theorem to find the distance between two points in a coordinate system.

Plane and solid geometry

10. Draw (freehand, with ruler and protractor, and with technology) geometric shapes from given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the triangle is uniquely defined, ambiguously defined or nonexistent.
11. Understand that slicing a three-dimensional figure with a plane produces a two-dimensional figure. Describe plane sections of right rectangular prisms and right rectangular pyramids.
12. Use hands-on activities to demonstrate and describe properties of: parallel lines in space, the line perpendicular to a given line through a given point, lines perpendicular to a given plane, lines parallel to a given plane, the plane or planes passing through three given points, and the plane perpendicular to a given line at a given point.

Statistics and Probability 8-SP

Patterns of association in bivariate data

1. Understand that scatter plots for bivariate measurement data may reveal patterns of association between two quantities.
2. Construct and interpret scatter plots for bivariate measurement data. Describe patterns such as clustering, outliers, positive or negative association, linear association, nonlinear association.
3. Understand that a straight line is a widely used model for exploring relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.
4. Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.*
5. Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables

collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. *For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?*

Mathematics Standards for High School

Where is the College-and-Career-Readiness line drawn?

The high school standards specify the mathematics that all students should learn in order to be college and career ready. The high school standards also describe additional mathematics that students should learn to pursue careers and majors in science, technology, engineering and mathematics (STEM) fields. Other forms of advanced work are possible (for example in discrete mathematics or advanced statistics) and can be eventually added to the standards.

Standards beyond the college and career readiness level that are necessary for STEM careers are prefixed with a symbol STEM, as in this example:

> STEM Graph complex numbers in polar form and interpret arithmetic operations on complex numbers geometrically.

Any standard without this tag is understood to be in the common core mathematics curriculum for all students.

How are the high school standards organized?

The high school standards are listed in conceptual categories, as shown in the Table below. **Appendix A (online) contains drafts of model course descriptions based on these standards**. Conceptual categories portray a coherent view of core high school mathematics; a student's work with Functions, for example, crosses a number of traditional course boundaries, potentially up through and including Calculus.

Conceptual Organization of the High School Standards

CCRS Draft September 17th	High School Standards Draft March 10
Number Quantity	Number and Quantity
Expressions Equations Coordinates	Algebra
Functions	Functions
Geometry	Geometry
Statistics Probability	Statistics and Probability
Modeling	Modeling**

* Standards formerly appearing under Coordinates now appear under other headings.

** Making mathematical models is now a Standard for Mathematical Practice. Standards formerly appearing under Modeling are now distributed under other major headings. High school standards with relevance to modeling are flagged with a (★) symbol. A narrative description of modeling remains in the high school standards, but there are no specific standard statements in that narrative description.

Mathematics | High School—Number and Quantity

Numbers and Number Systems. During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, "number" means "counting number": 1, 2, 3, … Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 7, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

Students sometimes have difficulty accepting new kinds of numbers when these differ in appearance and properties from those of a familiar system. For example, students might decide that complex numbers are not numbers because they are not written with numerical digits, or because they do not describe positive or negative quantities. Indeed, this ascent through number systems makes it fair to ask: what does the word *number* mean that it can mean all of these things? One possible answer is that a number is something that can be used to do mathematics: calculate, solve equations, or represent measurements. Historically, number systems have been extended when there is an intellectual or practical benefit in using the new numbers to solve previously insoluble problems.[1]

Although the referent of "number" changes, the four operations stay the same in important ways. The commutative, associative, and distributive laws extend the properties of operations to the integers, rational numbers, real numbers, and complex numbers. The inverse relationships between addition and subtraction, and multiplication and division are maintained in these larger systems.

Calculators are useful in this strand to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

Quantities. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, volume, and so forth. In high school, students encounter novel situations in which they themselves must conceive the attributes of interest. Such a conceptual process might be called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, who must conceptualize relevant attributes and create or choose suitable metrics in which to measure them.

Content Outline

The Real Number System

Quantities

The Complex Number System

Vector Quantities and Matrices

[1] See Harel, G., "A Standpoint of Research on Middle/Higher Number and Quantity," a research review provided for the Common Core State Standards Initiative.

The Real Number System N-RN

1. Understand that the laws of exponents for positive integer exponents follow from an understanding of exponents as indicating repeated multiplication, and from the associative law for multiplication.
2. Understand that the definition of the meaning of zero, positive rational, and negative exponents follows from extending the laws of exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, since* $(5^{1/3})^3 = 5^{(1/3)\cdot 3} = 5^1 = 5$, $5^{1/3}$ *is a cube root of* 5.
3. Understand that sums and products of rational numbers are rational.
4. Understand that the sum of a rational number and an irrational number is irrational, and that the product of a nonzero rational number and an irrational number is irrational.
5. Rewrite expressions using the laws of exponents. *For example,* $(5^{1/2})^3 = 5^{3/2}$ *and* $1/5 = 5^{-1}$.

Quantities★ N-Q

1. Understand that the magnitude of a quantity is independent of the unit used to measure it. *For example, the density of a liquid does not change when it is measured in another unit. Rather, its measure changes. The chosen unit "measures" the quantity by giving it a numerical value ("the density of lead is 11.3 times that of water").*
2. Use units as a way to understand problems and to guide the solution of multi-step problems, involving, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game.
3. Define metrics for the purpose of descriptive modeling. *For example, find a good measure of overall highway safety; propose and debate measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled.*
4. Add, subtract, multiply, and divide numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.
5. Use and interpret quantities and units correctly in algebraic formulas.
6. Use and interpret quantities and units correctly in graphs and data displays (function graphs, data tables, scatter plots, and other visual displays of quantitative information). Generate graphs and data displays using technology.

The Complex Number System N-CN

1. Understand that the relation $i^2 = -1$ and the commutative, associative, and distributive laws can be used to calculate with complex numbers.
2. STEM Understand that polynomials can be factored over the complex numbers, e.g., as in $x^2 + 4 = (x + 2i)(x - 2i)$.
3. STEM Understand that complex numbers can be visualized on the complex plane. Real numbers correspond to points on the horizontal (real) axis, and imaginary numbers to points on the vertical axis.
4. STEM Understand that on the complex plane, arithmetic of complex numbers can be interpreted geometrically: addition is analogous to vector addition, and multiplication can be understood as rotation and dilation about the origin. Complex conjugation is reflection across the real axis.
5. STEM Understand that on the complex plane, as on the real line, the distance between numbers is the absolute value of the difference, and the midpoint of a segment is the average of the numbers at its endpoints.
6. Add, subtract, and multiply complex numbers.
7. STEM Find the conjugate of a complex number; use conjugates to find absolute values and quotients of complex numbers.
8. STEM Solve quadratic equations with real coefficients that have complex solutions.
9. STEM Graph complex numbers in rectangular form.
10. STEM Graph complex numbers in polar form and interpret arithmetic operations on complex numbers geometrically.
11. STEM Explain why the rectangular and polar forms of a complex number represent the same number.

★ Standard with close connection to modeling.

1. STEM Understand that vector quantities have both magnitude and direction. Vector quantities are typically represented by directed line segments. The magnitude of a vector **v** is commonly denoted $|\mathbf{v}|$, $||\mathbf{v}||$, or v.
2. STEM Understand that vectors are determined by the coordinates of their initial and terminal points, or by their components.
3. STEM Understand that vectors can be added end-to-end, component-wise, or by the parallelogram rule. The magnitude of a sum of two vectors is typically not the sum of the magnitudes.
4. STEM Understand that a vector **v** can be multiplied by a real number c (called a scalar in this context) to form a new vector $c\mathbf{v}$ with magnitude $|c|v$. When $|c|v \neq 0$, the direction of $c\mathbf{v}$ is either along **v** (for $c > 0$) or against **v** (for $c < 0$). Scalar multiplication can be shown graphically by scaling vectors and possibly reflecting them in the origin; scalar multiplication can also be performed component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
5. STEM Understand that vector subtraction $\mathbf{v} - \mathbf{w}$ is defined as $\mathbf{v} + (-\mathbf{w})$. Two vectors can be subtracted graphically by connecting the tips in the appropriate order.
6. STEM Understand that matrices can be multiplied by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled. Matrices of the same dimensions can be added or subtracted. Matrices with compatible dimensions can be multiplied. Unlike multiplication of numbers, matrix multiplication is not a commutative operation, but still satisfies the associative and distributive laws.
7. STEM Understand that a vector, when regarded as a matrix with one column, can be multiplied by a matrix of suitable dimensions to produce another vector. A 2×2 matrix can be viewed as a transformation of the plane.
8. STEM Understand that a system of linear equations can be represented as a single matrix equation in a vector variable.
9. STEM Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
10. STEM Perform basic vector operations (addition, subtraction, scalar multiplication) both graphically and algebraically.
11. STEM Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
12. STEM Solve problems involving velocity and quantities that can be represented by vectors.★
13. STEM Add, subtract, and multiply matrices of appropriate dimensions.
14. STEM Use matrices to store and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
15. STEM Represent systems of linear equations as matrix equations.
16. STEM Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension greater than 3×3).

★ Standard with close connection to modeling.

Mathematics | High School—Algebra

Expressions. An expression is a description of a computation on numbers and symbols that represent numbers, using arithmetic operations and the operation of raising a number to rational exponents. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by deductions from the commutative, associative, and distributive laws and the inverse relationships between the four operations, and the conventions of algebraic notation. These extend what students have learned about arithmetic expressions in K–8 to expressions that involve exponents, radicals, and representations of real numbers, and, for STEM-intending students, complex numbers.

At times, an expression is the result of applying operations to simpler expressions. Viewing such an expression by singling out these simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a CAS environment can be used to experiment with algebraic expressions, perform complex algebraic manipulations,and understand how algebraic manipulations behave.

Equations and inequalities. An equation is a statement that two expressions are equal. Solutions to an equation are numbers that make the equation true when assigned to the variables in it. If the equation is true for all numbers, then it is called an identity; identities are often discovered by using the laws of arithmetic or the laws of exponents to transform one expression into another.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be graphed in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively transforming it into one or more simpler equations. The process is governed by deductions based on the properties of equality. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, stimulating the extension of that system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ((b_1+b_2)/2)h$, can be solved for h using the same deductive process.

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

Connections to Functions and Modeling. Expressions can define functions, and equivalent expressions define the same function. Equations in two variables may also define functions. Asking when two functions have the same value leads to an equation; graphing the two functions allows for the approximate solution of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.

Content Outline

Seeing Structure in Expressions

Arithmetic with Polynomials and Rational Expressions

Creating Equations that Describe Numbers or Relationships

Reasoning with Equations and Inequalities

Seeing Structure in Expressions A-SSE

1. Understand that different forms of an expression may reveal different properties of the quantity in question; a purpose in transforming expressions is to find those properties. *Examples: factoring a quadratic expression reveals the zeros of the function it defines, and putting the expression in vertex form reveals its maximum or minimum value; the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*
2. Understand that complicated expressions can be interpreted by viewing one or more of their parts as single entities.
3. Interpret an expression that represents a quantity in terms of the context. *Include interpreting parts of an expression, such as terms, factors and coefficients.*★
4. Factor, expand, and complete the square in quadratic expressions.
5. See expressions in different ways that suggest ways of transforming them. *For example, see* $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.
6. Rewrite expressions using the laws of exponents. *For example,* $(x^{1/2})^3 = x^{3/2}$ *and* $1/x = x^{-1}$.
7. Use the laws of exponents to interpret expressions for exponential functions, recognizing positive rational exponents as indicating roots of the base and negative exponents as indicating the reciprocal of a power. *For example, identify the per unit percentage change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and conclude whether it represents exponential growth or decay. Recognize that any nonzero number raised to the zero power is 1, for example,* $12(1.05)^0 = 12$. *Avoid common errors such as confusing* $6(1.05)^t$ *with* $(6 \cdot 1.05)^t$ *and* $5(0.03)^t$ *with* $5(1.03)^t$.
8. STEM Prove the formula for the sum of a geometric series, and use the formula to solve problems.

Arithmetic with Polynomials and Rational Expressions A-APR

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication.
2. Understand that polynomial identities become true statements no matter which real numbers are substituted. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
3. Understand the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
4. STEM Understand that the Binomial Theorem gives the expansion of $(x + a)^n$ in powers of x for a positive integer n and a real number a, with coefficients determined for example by Pascal's Triangle. The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.
5. STEM Understand that rational expressions are quotients of polynomials. They form a system analogous to the rational numbers, closed under division by a nonzero rational function.
6. Add, subtract and multiply polynomials.
7. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the polynomial.
8. Transform simple rational expressions using the commutative, associative, and distributive laws, and the inverse relationship between multiplication and division.
9. Divide a polynomial $p(x)$ by a divisor of the form $x - a$ using long division.
10. STEM Identify zeros and asymptotes of rational functions, when suitable factorizations are available, and use the zeros and asymptotes to construct a rough graph of the function.
11. STEM Divide polynomials, using long division for linear divisors and long division or a computer algebra system for higher degree divisors.

Creating Equations That Describe Numbers or Relationships A-CED*

1. Understand that equations in one variable are often created to describe properties of a specific but unknown number.
2. Understand that equations in two or more variables that represent a relationship between quantities can be built by experimenting with specific numbers in the relationship.
3. Write equations and inequalities that specify an unknown quantity or to express a relationship between two or more quantities. Use the equations and inequalities to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

★ Standard with close connection to modeling.

4. Rearrange formulas to highlight a quantity of interest. *For example, transform Ohm's law $V = IR$ to highlight resistance R; in motion with constant acceleration, transform $v_{f,x}^2 - v_{i,x}^2 = 2a_x(x_f - x_i)$ to highlight the change in position along the x-axis, $x_f - x_i$.*

Reasoning with Equations and Inequalities A-REI

1. Understand that to solve an equation algebraically, one makes logical deductions from the equality asserted by the equation, often in steps that replace it with a simpler equation whose solutions include the solutions of the original one.
2. Understand that the method of completing the square can transform any quadratic equation in x into an equivalent equation of the form $(x - p)^2 = q$. This leads to the quadratic formula.
3. Understand that given a system of two linear equations in two variables, adding a multiple of one equation to another produces a system with the same solutions. This principle, combined with principles already encountered with equations in one variable, allows for the simplification of systems.
4. Understand that the graph of an equation in two variables is the set of its solutions plotted in the coordinate plane, often forming a curve or a line.
5. Understand that solutions to two equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
6. Understand that the solutions to a linear inequality in two variables can be graphed as a half-plane (excluding the boundary in the case of a strict inequality).
7. Understand that solutions to several linear inequalities in two variables correspond to points in the intersection of the regions in the plane defined by the solutions to the inequalities.
8. Understand that equations and inequalities can be viewed as constraints in a problem situation, e.g., inequalities describing nutritional and cost constraints on combinations of different foods.★
9. STEM Understand that the relationship between an invertible function f and its inverse function can be used to solve equations of the form $f(x) = c$.
10. Solve simple rational and radical equations in one variable, noting and explaining extraneous solutions.
11. Solve linear equations in one variable, including equations with coefficients represented by letters.
12. Solve quadratic equations in one variable. *Include methods such as inspection (e.g. for $x^2 = 49$), square roots, completing the square, the quadratic formula and factoring. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.*
13. Solve equations $f(x) = g(x)$ approximately by finding the intersections of the graphs of $f(x)$ and $g(x)$, e.g. using technology to graph the functions. *Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, exponential, and logarithmic functions.*
14. Solve linear inequalities in one variable and graph the solution set on a number line.
15. Solve systems of linear equations algebraically and graphically, focusing on pairs of linear equations in two variables.
16. Solve algebraically a simple system consisting of one linear equation and one quadratic equation in two variables; for example, find points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.
17. Graph the solution set of a system of linear inequalities in two variables.
18. In modeling situations, represent constraints by systems of equations and/or inequalities, and interpret solutions of these systems as viable or non-viable options in the modeling context.★
19. In the context of exponential models, solve equations of the form $a\,b^{cx} = d$ where a, c, and d are specific numbers and the base b is 2, 10, or e.★
20. STEM Relate the properties of logarithms to the laws of exponents and solve equations involving exponential functions.
21. STEM Use inverse functions to solve equations of the form $a\sin(bx + c) = d$, $a\cos(bx + c) = d$, and $a\tan(bx + c) = d$.

★ Standard with close connection to modeling.

Mathematics | High School—Functions

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because nature and society are full of dependencies between quantities, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city"; or by an algebraic expression like $f(x) = a + bx$. The graph of a function is often a useful way of visualizing the relationship the function models, and manipulating a mathematical expression for a function can throw light on the function's properties. Graphing technology and spreadsheets are also useful tools in the study of functions.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a CAS can be used to experiment with properties of the functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling and Coordinates. Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.

Content Outline

Interpreting Functions

Building Functions

Linear, Quadratic, and Exponential Models

Trigonometric Functions

Limits and Continuity†

Differential Calculus†

Applications of Derivatives†

Integral Calculus†

Applications of Integration†

Infinite Series†

† Specific standards for calculus domains are not listed.

Interpreting Functions A-IF

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x.
2. Understand that functions of a single variable have key characteristics, including: zeros; extreme values; average rates of change (over intervals); intervals of increasing, decreasing and/or constant behavior; and end behavior.
3. Understand that a function defined by an expression may be written in different but equivalent forms, which can reveal different properties of the function.
4. Use function notation and evaluate functions for inputs in their domains.
5. Describe qualitatively the functional relationship between two quantities by reading a graph (e.g., where the function is increasing or decreasing, what its long-run behavior appears to be, and whether it appears to be periodic).★
6. Sketch a graph that exhibits the qualitative features of a function that models a relationship between two quantities.★
7. Compare properties of two functions represented in different ways (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, draw conclusions about the graph of a quadratic function from its algebraic expression.*
8. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★
9. Describe the qualitative behavior of functions presented in graphs and tables. *Identify: intercepts; intervals where the function is increasing, decreasing, positive or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
10. Use technology to exhibit the effects of parameter changes on the graphs of linear, power, quadratic, square root, cube root, and polynomial functions, and simple rational, exponential, logarithmic, sine, cosine, absolute value, and step functions.★
11. Transform quadratic polynomials algebraically to reveal different features of the function they define, such as zeros, extreme values, and symmetry of the graph.

Building Functions A-BF

1. Understand that functions can be described by specifying an explicit expression, a recursive process or steps for calculation.
2. Understand that sequences are functions whose domain is a subset of the nonnegative integers.
3. STEM Understand that composing a function f with a function g creates a new function called the composite function—for an input number x, the output of the composite function is $f(g(x))$.
4. STEM Understand that the inverse of an invertible function "undoes" what the function does; that is, composing the function with its inverse in either order returns the original input. One can sometimes produce an invertible function from a non-invertible function by restricting the domain (e.g., squaring is not an invertible function on the real numbers, but squaring is invertible on the nonnegative real numbers).
5. Write a function that describes a relationship between two quantities, for example by varying parameters in and combining standard function types (such as linear, quadratic or exponential functions). Experiment with parameters and illustrate an explanation of the behavior of the function when parameters vary using technology.★
6. Solve problems involving linear, quadratic, and exponential functions.★
7. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.
8. Generate an arithmetic or geometric sequence given a recursive rule for the sequence.★
9. As a way to describe routine modeling situations, write arithmetic and geometric sequences both recursively and in closed form, and translate between the two forms.★
10. STEM Evaluate composite functions and compose functions symbolically.
11. STEM Read values of an inverse function from a graph or a table, given that the function has an inverse.
12. STEM For linear or simple exponential functions, find a formula for an inverse function by solving an equation.
13. STEM Verify symbolically by composition that one function is the inverse of another.

Linear, Quadratic, and Exponential Models A-LQE

1. Understand that a linear function, defined by $f(x) = mx + b$ for some constants m and b, models a situation in which a quantity changes at a constant rate, m, relative to another.★
2. Understand that quadratic functions have maximum or minimum values and can be used to model problems with optimum solutions.★
3. Understand that an exponential function, defined by $f(x) = ab^x$ or by $f(x) = a(1 + r)^x$ for some constants a, $b > 0$ and $r > -1$, models a situation where a quantity grows or decays by a constant factor or a constant percentage change over each unit interval.★
4. Understand that linear functions grow by equal differences over equal intervals; exponential functions grow by equal factors over equal intervals.★
5. Understand that in an arithmetic sequence, differences between consecutive terms form a constant sequence, and second differences are zero. Conversely, if the second differences are zero, the sequence is arithmetic. Arithmetic sequences can be seen as linear functions.★
6. Understand that in a sequence that increases quadratically (e.g., $a_n = 3n^2 + 2n + 1$), differences between consecutive terms form an arithmetic sequence, and second differences form a constant sequence. Conversely, if the second differences form a constant sequence with nonzero value, the sequence increases quadratically.★
7. Understand that in a geometric sequence, ratios of consecutive terms are all the same.★
8. Understand that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★
9. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
10. Construct a function to describe a linear relationship between two quantities. Determine the rate of change and constant term of a linear function from a graph, a description of a relationship, or from two (x, y) values (include reading these from a table).★
11. Use quadratic functions to model problems, e.g., in situations with optimum solutions.★
12. Construct an exponential function in the form $f(x) = a(1 + r)^x$ or $f(x) = ab^x$ to describe a relationship in which one quantity grows with respect to another at a constant percent growth rate or a with a constant growth factor.★
13. Interpret the rate of change and constant term of a linear function or sequence in terms of the situation it models, and in terms of its graph or a table of values.★
14. Calculate and interpret the growth factor for an exponential function (presented symbolically or as a table) given a fixed time interval. Estimate the growth factor from a graph.★
15. Recognize a quantitative relationship as linear, exponential, or neither from description of a situation.★
16. Compare quantities increasing exponentially to quantities increasing linearly or as a polynomial function.★

Trigonometric Functions F-TF

1. STEM Understand that the unit circle in the coordinate plane enables one to define the sine, cosine, and tangent functions for real numbers.
2. STEM Understand that trigonometric functions are periodic by definition, and sums and products of functions with the same period are periodic.
3. STEM Understand that restricting trigonometric functions to a domain on which they are always increasing or always decreasing allows for the construction of an inverse function.
4. STEM Revisit trigonometric functions and their graphs in terms of radians.
5. STEM Use the unit circle to determine geometrically the values of sine, cosine, tangent for integer multiples of $\pi/4$ and $\pi/6$.
6. STEM Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.
7. STEM Solve simple trigonometric equations formally using inverse trigonometric functions and evaluate the solutions numerically using technology. *Solving trigonometric equations by means of the quadratic formula is optional.*

Limits and Continuity† F-LC

★ Standard with close connection to modeling.

† Specific standards for calculus domains are not listed.

Differential Calculus† F-DC

Applications of Derivatives† F-AD

Integral Calculus† F-IC

Applications of Integration† F-AI

Infinite Series† F-IS

Mathematics | High School—Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

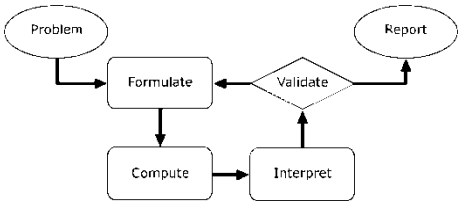
A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Risk situations, like extreme sports, pandemics and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.



The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those than represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then, either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution

or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, CAS environments, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Mathematics | High School—Statistics and Probability*

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model. One begins to make a probability model by listing or describing the possible outcomes (the sample space) and assigning probabilities. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the additive and multiplicative laws of probability. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, functional models, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

Connections to Functions and Modeling. Functional models may be used to approximate data; if the data are approximately linear, the relationship may be modeled with a regression line and the strength and direction of such a relationship may be expressed through a correlation coefficient.

Content Outline

Summarizing Categorical and Measurement Data

Probability Models

Independently Combined Probability Models

Making Inferences and Justifying Conclusions Drawn from Data

Conditional Probability and the Laws of Probability

Experimenting and Simulating to Model Probabilities

Using Probability to Make Decisions

* Most or all of the standards in Statistics and Probability have a close connection to modeling.

Summarizing Categorical and Quantitative Data S-SI

1. Understand that statistical methods take variability into account to support making informed decisions based on data collected to answer specific questions.
2. Understand that visual displays and summary statistics condense the information in data sets into usable knowledge.
3. Understand that patterns of association or relationships between variables may emerge through careful analysis of multi-variable data.
4. Summarize comparative or bivariate categorical data in two-way frequency tables. Interpret joint, marginal and conditional relative frequencies in the context of the data, recognizing possible associations and trends in bivariate categorical data.
5. Compare data on two or more count or measurement variables by using plots on the real number line (dot plots, histograms, and box plots). Use statistics appropriate to the shape of the data distribution to summarize center (median, mean) and spread (interquartile range, standard deviation) of the data sets. Interpret changes in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
6. Represent bivariate quantitative data on a scatter plot and describe how the variables are related.
7. Fit a linear function for scatter plots that suggest a linear association. Informally assess the fit of the model function by plotting and analyzing residuals.
8. Use a model function fitted to the data to solve problems in the context of the data, interpreting the slope (rate of change) and the intercept (constant term).
9. Compute (using technology) and interpret the correlation coefficient for a linear relationship between variables.
10. Distinguish between correlation and causation.

Probability Models S-PM

1. Understand that in a probability model, individual outcomes have probabilities that sum to 1. When outcomes are categorized, the probability of a given type of outcome is the sum of the probabilities of all the individual outcomes of that type.
2. Understand that uniform probability models are useful models for processes such as (i) the selection of a person from a population; (ii) the selection of a number in a lottery; (iii) any physical situation in which symmetry suggests that different individual outcomes are equally likely.
3. Understand that two different empirical probability models for the same process will rarely assign exactly the same probability to a given type of outcome. But if the data sets are large and the methods used to collect the data for the two data sets are consistent, the agreement between the models is likely to be reasonably good.
4. Understand that a (theoretical) uniform probability model may be judged by comparing it to an empirical probability model for the same process. If the theoretical assumptions are appropriate and the data set is large, then the two models should agree approximately. If the agreement is not good, then it may be necessary to modify the assumptions underlying the theoretical model or look for factors that might have affected the data used to create the empirical model.
5. Use a uniform probability model to compute probabilities for a process involving uncertainty, including the random selection of a person from a population and physical situations where symmetry suggests that different individual outcomes are equally likely.
   a. List the individual outcomes to create a sample space.
   b. Label the individual outcomes in the sample space to reflect important characteristics or quantities associated with them.
   c. Determine probabilities of individual outcomes, and determine the probability of a type or category of outcome as the fraction of individual outcomes it includes.
6. Generate data by sampling, repeated experimental trials, and simulations. Record and appropriately label such data, and use them to construct an empirical probability model. Compute probabilities in such models.
7. Compare probabilities from a theoretical model to probabilities from a corresponding empirical model for the same situation. If the agreement is not good, explain possible sources of the discrepancies.

Independently Combined Probability Models S-IPM

1. Understand that to describe a pair of random processes (such as tossing a coin and rolling a number cube), or one random process repeated twice (such as randomly selecting a student in the class on two different days), two probability models can be combined into a single model.

a. The sample space for the combined model is formed by listing all possible ordered pairs that combine an individual outcome from the first model with an individual outcome from the second. Each ordered pair is an individual outcome in the combined model.

b. The total number of individual outcomes (ordered pairs) in the combined model is the product of the number of individual outcomes in each of the two original models.

2. Understand that when two probability models are combined independently, the probability that one type of outcome in the first model occurs together with another type of outcome in the second model is the product of the two corresponding probabilities in the original models (the Multiplication Rule).

3. Combine two uniform models independently to compute probabilities for a pair of random processes (e.g., flipping a coin twice, selecting one person from each of two classes).

   a. Use organized lists, tables and tree diagrams to represent the combined sample space.

   b. Determine probabilities of ordered pairs in the combined model, and determine the probability of a particular type or category of outcomes in the combined model, as the fraction of ordered pairs corresponding to it.

4. For two independently combined uniform models, use the Multiplication Rule to determine probabilities.

Making Inferences and Justifying Conclusions S-IC

1. Understand that statistics is a process for making inferences about population parameters based on a sample from that population; randomness is the foundation for statistical inference.
2. Understand that the design of an experiment or sample survey is of critical importance to analyzing the data and drawing conclusions.
3. Understand that simulation-based techniques are powerful tools for making inferences and justifying conclusions from data.
4. Use probabilistic reasoning to decide if a specified model is consistent with results from a given data-generating process. (For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?)
5. Recognize the purposes of and differences among sample surveys, experiments and observational studies; explain how randomization relates to each.
6. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.
7. Use data from a randomized experiment to compare two treatments; justify significant differences between parameters through the use of simulation models for random assignment.
8. Evaluate reports based on data.

Conditional Probability and the Laws of Probability S-CP

1. Understand that events are subsets of a sample space; often, events of interest are defined by using characteristics (or categories) of the sample points, or as unions, intersections, or complements thereof ("and," "or," "not"). A sample point may belong to several events (categories).
2. Understand that if A and B are two events, then in a uniform model the conditional probability of A given B, denoted by $P(A \mid B)$, is the fraction of B's sample points that also lie in A.
3. Understand that the laws of probability allow one to use known probabilities to determine other probabilities of interest.
4. Compute probabilities by constructing and analyzing sample spaces, representing them by tree diagrams, systematic lists, and Venn diagrams.
5. Use the laws of probability to compute probabilities.
6. Apply concepts such as intersections, unions and complements of events, and conditional probability and independence to define or analyze events, calculate probabilities and solve problems.
7. Construct and interpret two-way tables to show probabilities when two characteristics (or categories) are associated with each sample point. Use a two-way table to determine conditional probabilities.★
8. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.★
9. Use permutations and combinations to compute probabilities of compound events and solve problems.

★ Standard with close connection to modeling.

Experimenting and Simulating to Model Probabilities S-ES

1. Understand that sets of data obtained from surveys, simulations or other means can be used as probability models, by treating the data set itself as a sample space, in which the sample points are the individual pieces of data.
2. Understand that the probability of an outcome can be interpreted as an assertion about the long-run proportion of the outcome's occurrence if the random experiment is repeated a large number of times.
3. Calculate experimental probabilities by performing simulations or experiments involving a probability model and using relative frequencies of outcomes.
4. Compare the results of simulations with predicted probabilities. When there are substantial discrepancies between predicted and observed probabilities, explain them.
5. Use the mean and standard deviation of a data set to fit it to a normal distribution (bell-shaped curve) and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets and tables to estimate areas under the normal curve.

Using Probability to Make Decisions S-MD

1. Understand that the expected value of a random variable is the weighted average of its possible values, with weights given by their respective probabilities.
2. Understand that when the possible outcomes of a decision can be assigned probabilities and payoff values, the decision can be analyzed as a random variable with an expected value, e.g., of an investment.
3. Calculate expected value, e.g. to determine the fair price of an investment.
4. Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).
5. Evaluate and compare two investments or strategies with the same expected value, where one investment or strategy is safer than the other.
6. Evaluate and compare two investments or strategies, where one investment or strategy is safer but has lower expected value. Include large and small investments, and situations with serious consequences.
7. Analyze decisions and strategies using probability concepts (e.g. product testing, medical testing, pulling a hockey goalie at the end of a game).

Mathematics | High School—Geometry

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Understanding the attributes of geometric objects often relies on measurement: a circle is a set of points in a plane at a fixed distance from a point; a cube is bounded by six squares of equal area; when two parallel lines are crossed by a transversal, pairs of corresponding angles are congruent.

The concepts of congruence, similarity and symmetry can be united under the concept of geometric transformation. Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent. Applying a scale transformation to a geometric figure yields a similar figure. The transformation preserves angle measure, and lengths are related by a constant of proportionality.

The definitions of sine, cosine and tangent for acute angles are founded on right triangle similarity, and, with the Pythagorean theorem, are fundamental in many real-world and theoretical situations.

Coordinate geometry is a rich field for exploration. How does a geometric transformation such as a translation or reflection affect the coordinates of points? How is the geometric definition of a circle reflected in its equation? Coordinates can describe locations in three dimensions and extend the use of algebraic techniques to problems involving the three-dimensional world we live in.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same was as CAS environments allow them to experiment with algebraic phenomena.

Connections to Equations and Inequalities. The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling and proof.

Content Outline

Congruence

Similarity, Right Triangles, and Trigonometry

Circles

Expressing Geometric Properties with Equations

Trigonometry of General Triangles

Geometric Measurement and Dimension

Modeling with Geometry

Congruence G-CO

1. Understand that two geometric figures are congruent if there is a sequence of rigid motions (rotations, reflections, translations) that carries one onto the other. This is the principle of superposition.
2. Understand that criteria for triangle congruence are ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent.
3. Understand that criteria for triangle congruence (ASA, SAS, and SSS) can be established using rigid motions.
4. Understand that geometric diagrams can be used to test conjectures and identify logical errors in fallacious proofs.
5. Know and use (in reasoning and problem solving) definitions of angles, polygons, parallel, and perpendicular lines, rigid motions, parallelograms and rectangles.
6. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; two lines parallel to a third are parallel to each other; points on a perpendicular bisector of a segment are exactly those equidistant from the segment's endpoints.*
7. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°, base angles of isosceles triangles are congruent, the triangle inequality, the longest side of a triangle faces the angle with the greatest measure and vice-versa, the exterior-angle inequality, and the segment joining midpoints of two sides of a triangle parallel to the third side and half the length.*
8. Use and prove properties of and relationships among special quadrilaterals: parallelogram, rectangle, rhombus, square, trapezoid and kite.
9. Characterize parallelograms in terms of equality of opposite sides, in terms of equality of opposite angles, and in terms of bisection of diagonals; characterize rectangles as parallelograms with equal diagonals.
10. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
11. Construct an equilateral triangle, a square and a regular hexagon inscribed in a circle.
12. Use two-dimensional representations to transform figures and to predict the effect of translations, rotations, and reflections.
13. Use two-dimensional representations to transform figures and to predict the effect of dilations.

Similarity, Right Triangles, and Trigonometry G-SRT

1. Understand that dilating a line produces a line parallel to the original. (In particular, lines passing through the center of the dilation remain unchanged.)
2. Understand that the dilation of a given segment is parallel to the given segment and longer or shorter in the ratio given by the scale factor. A dilation leaves a segment unchanged if and only if the scale factor is 1.
3. Understand that the assumed properties of dilations can be used to establish the AA, SAS, and SSS criteria for similarity of triangles.
4. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of sine, cosine, and tangent.
5. Understand that a line parallel to one side of a triangle divides the other two proportionally, and conversely.
6. Use triangle similarity criteria to solve problems and to prove relationships in geometric figures. *Include a proof of the Pythagorean theorem using triangle similarity.*
7. Use and explain the relationship between the sine and cosine of complementary angles.
8. Use sine, cosine, tangent, and the Pythagorean Theorem to solve right triangles[2] in applied problems.
9. STEM Give an informal explanation using successive approximation that a dilation of scale factor r changes the length of a curve by a factor of r and the area of a region by a factor of r^2.

Circles G-C

1. Understand that dilations can be used to show that all circles are similar.
2. Understand that here is a unique circle through three non-collinear points, or tangent to three non-concurrent lines.
3. Identify and define radius, diameter, chord, tangent, secant, and circumference.

[2] A right triangle has five parameters, its three lengths and two acute angles. Given a length and any other parameter, "solving a right triangle" means finding the remaining three parameters.

4. Identify and describe relationships among angles, radii, and chords. *Include the relationship between central, inscribed and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
5. Determine the arc lengths and the areas of sectors of circles, using proportions.
6. STEM Construct a tangent line from a point outside a given circle to the circle.
7. STEM Prove and use theorems about circles, and use these theorems to solve problems involving:
   a. Symmetries of a circle
   b. Similarity of a circle to any other
   c. Tangent line, perpendicularity to a radius
   d. Inscribed angles in a circle, relationship to central angles, and equality of inscribed angles
   e. Properties of chords, tangents, and secants as an application of triangle similarity.

Expressing Geometric Properties with Equations G-GPE

1. Understand that two lines with well-defined slopes are perpendicular if and only if the product of their slopes is equal to –1.
2. Understand that the equation of a circle can be found using its definition and the Pythagorean Theorem.
3. Understand that transforming the graph of an equation by reflecting in the axes, translating parallel to the axes, or applying a dilation to one of the axes correspond to substitutions in the equation. *For example, reflection in the y axis corresponds to* $(x,y) \rightarrow (-x,y)$*, translation vertically down by three units corresponds to* $(x,y) \rightarrow (x,y+3)$*, and dilating by a factor of 2 parallel to the x-axis corresponds to* $(x, y) \rightarrow (x/2, y)$.
4. STEM Understand that an ellipse is the set of all points whose distances from two fixed points (the foci) are a constant sum. The graph of $x^2/a^2 + y^2/b^2 = 1$ is an ellipse with foci on one of the axes.
5. STEM Understand that a parabola is the set of points equidistant from a fixed point (the focus) and a fixed line (the directrix). The graph of any quadratic function is a parabola, and all parabolas are similar.
6. STEM Understand that the formula $A = \pi ab$ for the area of an ellipse can be derived from the formula for the area of a circle.★
7. Use the slope criteria for parallel and perpendicular lines to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
8. Find the point on the segment between two given points that divides the segment in a given ratio.
9. Use coordinates to compute perimeters of polygons and areas for triangles and rectangles, e.g. using the distance formula.★
10. Decide whether a point with given coordinates lies on a circle defined by a given equation.
11. Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point (0, 2).
12. Complete the square to find the center and radius of a circle given by an equation.
13. STEM Find an equation for an ellipse given in the coordinate plane with major and minor axes parallel to the coordinate axes.
14. STEM Calculate areas of ellipses to solve problems.★

Trigonometry of General Triangles G-TGT

1. STEM Understand that the formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle can be derived by drawing an auxiliary line from a vertex perpendicular to the opposite side. Applying this formula in three different ways leads to the Law of Sines.
2. STEM Understand that the Law of Cosines generalizes the Pythagorean Theorem.
3. STEM Understand that the sine, cosine and tangent of the sum or difference of two angles can be expressed in terms of sine, cosine, and tangent of the angles themselves using the addition formulas.
4. STEM Understand that the Laws of Sines and Cosines embody the triangle congruence criteria, in that three pieces of information are usually sufficient to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that "Side-Side-Angle" is not a congruence criterion.
5. STEM Explain proofs of the Law of Sines and the Law of Cosines.

★ Standard with close connection to modeling.

6. STEM Use the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Geometric Measurement and Dimension G-GMD

1. Understand that the area of a decomposed figure is the sum of the areas of its components and is independent of the choice of dissection.
2. STEM Understand that lengths of curves and areas of curved regions can be defined using the informal notion of limit.
3. STEM Understand that Cavalieri's principle allows one to understand volume formulas informally by visualizing volumes as stacks of thin slices.
4. Find areas of polygons by dissecting them into triangles.
5. Explain why the volume of a cylinder is the area of the base times the height, using informal arguments.
6. For a pyramid or a cone, give a heuristic argument to show why its volume is one-third of its height times the area of its base.
7. Apply formulas and solve problems involving volume and surface area of right prisms, right circular cylinders, right pyramids, cones, spheres and composite figures.
8. STEM Identify cross-sectional shapes of slices of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.
9. STEM Use the behavior of length and area under dilations to show that the circumference of a circle is proportional to the radius and the area of a circle is proportional to the square of the radius. Identify the relation between the constants of proportionality with an informal argument involving dissection and recomposition of a circle into an approximate rectangle.

Modeling with Geometry G-MG

1. Understand that models of objects and structures can be built from a library of standard shapes; a single kind of shape can model seemingly different objects.★
2. Use geometric shapes, their measures and their properties to describe objects (e.g., modeling a tree trunk or a human torso or as a cylinder).★
3. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★
4. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy constraints or minimize cost; working with typographic grid systems based on ratios).★

★ Standard with close connection to modeling.

Glossary

Addition and subtraction within 10, 20, or 100. Addition or subtraction of whole numbers with whole number answers, and with sum or minuend at most 10, 20, or 100. Example: $8 + 2 = 10$ is an addition within 10, $14 - 5 = 9$ is a subtraction within 20, and $55 - 18 = 37$ is a subtraction within 100.

Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. Example: $3/4$ and $-3/4$ are additive inverses of one another because $3/4 + (-3/4) = (-3/4) + 3/4 = 0$.

Box plot. A method of visually displaying a distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data.[3]

Complex fraction. A fraction A/B where A and/or B are fractions.

Congruent. Two plane or solid figures are congruent if one can be obtained from the other by a sequence of rigid motions (rotations, reflections, and translations).

Counting on. A strategy for finding the number of objects in a group without having to count every member of the group. For example, if a stack of books is known to have 8 books and 3 more books are added to the top, it is not necessary to count the stack all over again; one can find the total by *counting on*—pointing to the top book and saying "eight," following this with "nine, ten, eleven. There are eleven books now."

Decade word. A word referring to a single-digit multiple of ten, as in *twenty*, *thirty*, *forty*, etc.

Dot plot. A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a line plot.[4]

Dilation. A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

Empirical probability model. A probablity model based on a data set for a random process in which the probability of a particular type or category of outcome equals the percentage of data points included in the category. Example: If a coin is tossed 10 times and 4 of the tosses are Heads, then the empirical probability of Heads in the empirical probability model is $4/10$ (equivalently 0.4 or 40%).

Equivalent fractions. Two fractions a/b and c/d that represent the same number.

Expanded form. A multidigit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. For example, $643 = 600 + 40 + 3$.

First quartile. For a data set with median M, the first quartile is the median of the data values less than M. Example: For the data set $\{1, 3, 6, 7, 10, 12, 14, 15, 22, 120\}$, the first quartile is 6.[5] *See also* median, third quartile, interquartile range.

Fraction. A number expressible in the form a/b where a is a whole number and b is a positive whole number. (The word *fraction* in these standards always refers to a nonnegative number.) *See also* rational number.

Independently combined probability models. Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

Integer. A number expressible in the form a or $-a$ for some whole number a.

Interquartile Range. A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set $\{1, 3, 6, 7, 10, 12, 14, 15, 22, 120\}$, the interquartile range is $15 - 6 = 9$. *See also* first quartile, third quartile.

Laws of arithmetic. See Table 3 in this Glossary.

Line plot. See *dot plot*.

Mean. A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list.[6] Example: For the data set $\{1, 3, 6, 7, 10, 12, 14, 15, 22, 120\}$, the mean is 21.

Mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. Example: For the data set $\{2, 3, 6, 7, 10, 12, 14, 15, 22, 120\}$, the mean absolute deviation is 20.

Median. A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list—or the mean of the two central values, if the list contains an even number of values. Example: For the data set $\{2, 3, 6, 7, 10, 12, 14, 15, 22, 90\}$, the median is 11.

---

3 Adapted from Wisconsin Department of Public Instruction, http://dpi.wi.gov/standards/mathglos.html, accessed March 2, 2010.

4 Adapted from Wisconsin Department of Public Instruction, *op. cit.*.

5 Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See Langford, E., "Quartiles in Elementary Statistics, *Journal of Statistics Education* Volume 14, Number 3 (2006),

6 To be more precise, this defines the *arithmetic mean*.

Multiplication and division within 100. Multiplication or division of whole numbers with whole number answers, and with product or dividend at most 100. Example: $72 \div 8 = 9$.

Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. Example: $3/4$ and $4/3$ are multiplicative inverses of one another because $3/4 \times 4/3 = 4/3 \times 3/4 = 1$.

Properties of equality. See Table 4 in this Glossary.

Properties of inequality. See Table 5 in this Glossary.

Properties of operations. Associativity and commutativity of addition and multiplication, distributivity of multiplication over addition, the additive identity property of 0, and the multiplicative identity property of 1.See Table 3 in this Glossary.

Probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, testing for a medical condition).

Rational number. A number expressible in the form a/b or $-a/b$ for some fraction a/b. The rational numbers include the integers.

Related fractions. Two fractions are said to be related if one denominator is a factor of the other.[7]

Rigid motion. A transformation of points in space consisting of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

Sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.

Scatter plot. A graph in the coordinate plane representing a set of bivariate data. For example, the heights and weights of a group of people could be displayed on a scatter plot.[8]

Similarity transformation. A rigid motion followed by a dilation.

Tape diagrams. Drawings that look like a segment of tape, used to illustrate number relationships. Also known as strip diagrams, bar models or graphs, fraction strips, or length models.

Teen number. A whole number that is greater than or equal to 11 and less than or equal to 19.

Third quartile. For a data set with median M, the third quartile is the median of the data values greater than M. Example: For the data set $\{2, 3, 6, 7, 10, 12, 14, 15, 22, 120\}$, the third quartile is 15. *See also* median, first quartile, interquartile range.

Uniform probability model. A probability model in which the individual outcomes all have the same probability ($1/N$ if there are N individual outcomes in the sample space). If a given type of outcome consists of M individual outcomes, then the probability of that type of outcome is M/N. Example: if a uniform probability model is used to model the process of randomly selecting a person from a class of 32 students, and if 8 of the students are left-handed, then the probability of randomly selecting a left-handed student is $8/32$ (equivalently $1/4$, 0.25 or 25%).

Whole numbers. The numbers 0, 1, 2, 3, ….

[7] See Ginsburg, Leinwand and Decker (2009), *Informing Grades 1-6 Mathematics Standards Development: What Can Be Learned from High-Performing Hong Kong, Korea, and Singapore?*, Table A1, p. A-5, grades 3 and 4.

[8] Adapted from Wisconsin Department of Public Instruction, *op. cit.*.

TABLE 1. Common addition and subtraction situations.[9]

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? $5 – 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 – ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? – 2 = 3$

	Total Unknown	Addend Unknown	Both Addends Unknown[10]
Put Together/ Take Apart[11]	Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5,\ 5 – 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5,\ 5 = 5 + 0$ $5 = 1 + 4,\ 5 = 4 + 1$ $5 = 2 + 3,\ 5 = 3 + 2$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare[12]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5,\ 5 – 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?,\ 3 + 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 – 3 = ?,\ ? + 3 = 5$

[9] Adapted from Box 2-4 of National Research Council (2009, op. cit., pp. 32, 33).
[10] These *take apart* situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes* or *results in* but always does mean *is the same number as*.
[11] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation especially for small numbers less than or equal to 10.
[12] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

TABLE 2. Common multiplication and division situations.[13]

	Unknown Product	**Group Size Unknown** (“How many in each group?” Division)	**Number of Groups Unknown** (“How many groups?” Division)
	3 × 6 = ?	**3 × ? = 18 and 18 ÷ 3 = ?**	**? × 6 = 18 and 18 ÷ 6 = ?**
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[14] Area[15]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

[13] The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.
[14] The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.
[15] Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

TABLE 3. The laws of arithmetic, including the properties of operations (identified with °). Here a, b and c stand for arbitrary numbers in a given number system. The laws of arithmetic apply to the rational number system, the real number system, and the complex number system.

°*Associative law of addition*	$(a + b) + c = a + (b + c)$
°*Commutative law of addition*	$a + b = b + a$
°*Additive identity property of 0*	$a + 0 = 0 + a = a$
Existence of additive inverses	For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0$.
°*Associative law of multiplication*	$(a \times b) \times c = a \times (b \times c)$
°*Commutative law of multiplication*	$a \times b = b \times a$
°*Multiplicative identity property of 1*	$a \times 1 = 1 \times a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$.
°*Distributive law of multiplication over addition*	$a \times (b + c) = a \times b + a \times c$

TABLE 4. The properties of equality. Here a, b and c stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.

TABLE 5. The properties of inequality. Here a, b and c stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.
If $a > b$ and $b > c$ then $a > c$.
If $a > b$, then $b < a$.
If $a > b$, then $-a < -b$.
If $a > b$, then $a \pm c > b \pm c$.
If $a > b$ and $c > 0$, then $a \times c > b \times c$.
If $a > b$ and $c < 0$, then $a \times c < b \times c$.
If $a > b$ and $c > 0$, then $a \div c > b \div c$.
If $a > b$ and $c < 0$, then $a \div c < b \div c$.

Sample of Works Consulted

Existing state standards documents.

Research summaries and briefs provided to the Working Group by researchers.

Mathematics documents from: Alberta, Canada; Belgium; China; Chinese Taipei; Denmark; England; Finland; Hong Kong; India; Ireland; Japan; Korea, New Zealand, Singapore; Victoria (British Columbia).

Adding it Up: Helping Children Learn Mathematics. National Research Council, Mathematics Learning Study Committee, 2001.

Benchmarking for Success: Ensuring U.S. Students Receive a World-Class Education. National Governors Association, Council of Chief State School Officers, and Achieve, Inc., 2008.

Crossroads in Mathematics (1995) and *Beyond Crossroads* (2006). American Mathematical Association of Two-Year Colleges (AMATYC).

Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence. National Council of Teachers of Mathematics, 2006.

Focus in High School Mathematics: Reasoning and Sense Making. National Council of Teachers of Mathematics. Reston, VA: NCTM.

Foundations for Success: The Final Report of the National Mathematics Advisory Panel. U.S. Department of Education: Washington, DC, 2008.

Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A PreK-12 Curriculum Framework.

How People Learn: Brain, Mind, Experience, and School. Bransford, J.D., Brown, A.L., and Cocking, R.R., eds. Committee on Developments in the Science of Learning, Commission on Behavioral and Social Sciences and Education, National Research Council, 1999.

Mathematics and Democracy, The Case for Quantitative Literacy, Steen, L.A. (ed.). National Council on Education and the Disciplines, 2001.

Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity. Cross, C.T., Woods, T.A., and Schweingruber, S., eds. Committee on Early Childhood Mathematics, National Research Council, 2009.

The Opportunity Equation: Transforming Mathematics and Science Education for Citizenship and the Global Economy. The Carnegie Corporation of New York and the Institute for Advanced Study, 2009. Online: http://www.opportunityequation.org/

Principles and Standards for School Mathematics. National Council of Teachers of Mathematics, 2000.

The Proficiency Illusion. Cronin, J., Dahlin, M., Adkins, D., and Kingsbury, G.G.; foreword by C.E. Finn, Jr., and M. J. Petrilli. Thomas B. Fordham Institute, 2007.

Ready or Not: Creating a High School Diploma That Counts. American Diploma Project, 2004.

A Research Companion to Principles and Standards for School Mathematics. National Council of Teachers of Mathematics, 2003.

Sizing Up State Standards 2008. American Federation of Teachers, 2008.

A Splintered Vision: An Investigation of U.S. Science and Mathematics Education. Schmidt, W.H., McKnight, C.C., Raizen, S.A., et al. U.S. National Research Center for the Third International Mathematics and Science Study, Michigan State University, 1997.

Stars By Which to Navigate? Scanning National and International Education Standards in 2009. Carmichael, S.B., W.S. Wilson, Finn, Jr., C.E., Winkler, A.M., and Palmieri, S. Thomas B. Fordham Institute, 2009.

Blum, W., Galbraith, P. L., Henn, H-W. and Niss, M. (Eds) *Applications and Modeling in Mathematics Education*, ICMI Study 14. Amsterdam: Springer.

Cobb and Moore, "Mathematics, Statistics, and Teaching," *Amer. Math. Monthly* 104(9), pp. 801-823, 1997.

Conley, D.T. *Knowledge and Skills for University Success*, 2008.

Conley, D.T. *Toward a More Comprehensive Conception of College Readiness*, 2007.

Cuoco, A., Goldenberg, E. P., and Mark, J., "Habits of Mind: An Organizing Principle for a Mathematics Curriculum," *Journal of Mathematical Behavior*, 15(4), 375-402, 1996.

Ginsburg, A., Leinwand, S., and Decker, K., "Informing Grades 1-6 Standards Development: What Can Be Learned from High-Performing Hong Kong, Korea, and Singapore?" American Institutes for Research, 2009.

Ginsburg et al., "What the United States Can Learn From Singapore's World-Class Mathematics System (and what Singapore can learn from the United States)," American Institutes for Research, 2005.

Ginsburg et al., "Reassessing U.S. International Mathematics Performance: New Findings from the 2003 TIMMS and PISA," American Institutes for Research, 2005.

Harel, G., "What is Mathematics? A Pedagogical Answer to a Philosophical Question," in R. B. Gold and R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics from the Perspective of Mathematicians*. Mathematical Association of America, 2008.

Howe, R., "From Arithmetic to Algebra."

Howe, R., "Starting Off Right in Arithmetic," http://math.arizona.edu/~ime/2008-09/MIME/BegArith.pdf.

Jordan, N. C., Kaplan, D., Ramineni, C., and Locuniak, M. N., "Early math matters: kindergarten number competence and later mathematics outcomes," *Dev. Psychol.* 45, 850–867, 2009.

Kilpatrick, J., Mesa, V., and Sloane, F., "U.S. Algebra Performance in an International Context," in Loveless (ed.), *Lessons Learned: What International Assessments Tell Us About Math Achievement*. Washington, D.C.: Brookings Institution Press, 2007.

Leinwand, S., and Ginsburg, A., "Measuring Up: How the Highest Performing state (Massachusetts) Compares to the Highest Performing Country (Hong Kong) in Grade 3 Mathematics," American Institutes for Research, 2009.

Niss, M., "Quantitative Literacy and Mathematical Competencies," in *Quantitative Literacy: Why Numeracy Matters for Schools and Colleges*, Madison, B. L., and Steen, L.A. (eds.), National Council on Education and the Disciplines. Proceedings of the National Forum on Quantitative Literacy held at the National Academy of Sciences in Washington, D.C., December 1-2, 2001.

Reys, B. (ed.), *The Intended Mathematics Curriculum as Represented in State-Level Curriculum Standards: Consensus or Confusion?* IAP-Information Age Publishing, 2006.

Schmidt, W., Houang, R., and Cogan, L., "A Coherent Curriculum: The Case of Mathematics," *American Educator*, Summer 2002, p. 4.

Schmidt, W.H. and Houang, R.T., "Lack of Focus in the Intended Mathematics Curriculum: Symptom or Cause?" in Loveless (ed.), *Lessons Learned: What International Assessments Tell Us About Math Achievement*. Washington, D.C.: Brookings Institution Press, 2007.

Wu, H., "Fractions, decimals, and rational numbers," 2007, http://math.berkeley.edu/~wu/ (March 19, 2008).

Wu, H., "Lecture Notes for the 2009 Pre-Algebra Institute," September 15, 2009.

Wu, H., "Preservice professional development of mathematics Teachers," http://math.berkeley.edu/~wu/pspd2.pdf.

Massachusetts Department of Education. Progress Report of the Mathematics Curriculum Framework Revision Panel, Massachusetts Department of Elementary and Secondary Education, 2009. www.doe.mass.edu/boe/docs/0509/item5_report.pdf.

ACT College Readiness Benchmarks™

ACT College Readiness Standards™

ACT National Curriculum Survey™

Adelman, C. *The Toolbox Revisited: Paths to Degree Completion From High School Through College*, 2006.

Advanced Placement Calculus, Statistics and Computer Science Course Descriptions. May 2009, May 2010. College Board, 2008.

Aligning Postsecondary Expectations and High School Practice: The Gap Defined (ACT: Policy Implications of the ACT National Curriculum Survey Results 2005-2006).

Condition of Education, 2004: Indicator 30, Top 30 Postsecondary Courses, U.S. Department of Education, 2004.

Condition of Education, 2007: High School Course-Taking. U.S. Department of Education, 2007.

Crisis at the Core: Preparing All Students for College and Work, ACT.

Achieve, Inc., Florida Postsecondary Survey, 2008.

Golfin, Peggy, et. al. CNA Corporation. *Strengthening Mathematics at the Postsecondary Level: Literature Review and Analysis*, 2005.

Camara, W.J., Shaw, E., and Patterson, B. (June 13, 2009). First Year English and Math College Coursework. College Board: New York, NY (Available from authors).

CLEP Precalculus Curriculum Survey: Summary of Results. The College Board, 2005.

College Board Standards for College Success: Mathematics and Statistics. College Board, 2006.

Miller, G.E., Twing, J., and Meyers, J. "Higher Education Readiness Component (HERC) Correlation Study." Austin, TX: Pearson.
On Course for Success: A Close Look at Selected High School Courses That Prepare All Students for College and Work, ACT.
Out of Many, One: Towards Rigorous Common Core Standards from the Ground Up. Achieve, 2008.
Ready for College and Ready for Work: Same or Different? ACT.
Rigor at Risk: Reaffirming Quality in the High School Core Curriculum, ACT.
The Forgotten Middle: Ensuring that All Students Are on Target for College and Career Readiness before High School, ACT.
Achieve, Inc., Virginia Postsecondary Survey, 2004.
ACT Job Skill Comparison Charts
Achieve, Mathematics at Work, 2008.
The American Diploma Project Workplace Study. National Alliance of Business Study, 2002.
Carnevale, Anthony and Desrochers, Donna. *Connecting Education Standards and Employment: Course-taking Patterns of Young Workers,* 2002.
Colorado Business Leaders Top Skills, 2006.
Hawai'i Career Ready Study: access to living wage careers from high school, 2007.
States' Career Cluster Initiative. *Essential Knowledge and Skill Statements*, 2008.
ACT WorkKeys Occupational Profiles™
Program for International Student Assessment (PISA), 2006.
Trends in International Mathematics and Science Study (TIMSS), 2007.
International Baccalaureate, Mathematics Standard Level, 2006.
University of Cambridge International Examinations: General Certificate of Secondary Education in Mathematics, 2009.
EdExcel, General Certificate of Secondary Education, Mathematics, 2009.
Blachowicz, Camille, and Peter Fisher. "Vocabulary Instruction." In *Handbook of Reading Research*, Volume III, edited by Michael Kamil, Peter Mosenthal, P. David Pearson, and Rebecca Barr, pp. 503-523. Mahwah, NJ: Lawrence Erlbaum Associates, 2000.
Gándara, Patricia, and Frances Contreras. *The Latino Education Crisis: The Consequences of Failed Social Policies.* Cambridge, Ma: Harvard University Press, 2009.
Moschkovich, Judit N. "Supporting the Participation of English Language Learners in Mathematical Discussions." *For the Learning of Mathematics* 19 (March 1999): 11-19.
Moschkovich, J. N. (in press). Language, culture, and equity in secondary mathematics classrooms. To appear in F. Lester & J. Lobato (Ed.), Teaching and Learning Mathematics: Translating Research to the Secondary Classroom, Reston, VA: NCTM.
Moschkovich, Judit N. "Examining Mathematical Discourse Practices," *For the Learning of Mathematics* 27 (March 2007): 24-30.
Moschkovich, Judit N. "Using Two Languages when Learning Mathematics: How Can Research Help Us Understand Mathematics Learners Who Use Two Languages?" *Research Brief and Clip*, National Council of Teachers of Mathematics, 2009 http://www.nctm.org/uploadedFiles/Research_News_and_Advocacy/Research/Clips_and_Briefs/Research_brief_12_Using_2.pdf. (accessed November 25, 2009).
Moschkovich, J.N. (2007) Bilingual Mathematics Learners: How views of language, bilingual learners, and mathematical communication impact instruction. In N. Nasir and P. Cobb (Eds.), *Diversity, Equity, and Access to Mathematical Ideas*. New York: Teachers College Press, 89-104.
Schleppegrell, M.J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly,* 23:139-159.
Individuals with Disabilities Education Act (IDEA), 34 CFR §300.34 (a). (2004).
Individuals with Disabilities Education Act (IDEA), 34 CFR §300.39 (b)(3). (2004).
Office of Special Education Programs, U.S. Department of Education. "IDEA Regulations: Identification of Students with Specific Learning Disabilities," 2006.
Thompson, S. J., Morse, A.B., Sharpe, M., and Hall, S., "Accommodations Manual: How to Select, Administer and Evaluate Use of Accommodations and Assessment for Students with Disabilities," 2nd Edition. Council of Chief State School Officers, 2005.